Radial Control Algorithm for Two Dimensional Manifold Computation

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Abstract: The Algorithms of manifold computation are still remained to be optimized. The low computing errors and high computing speed are always two incompatible factors. This paper takes the discrete system as study object and combines the manifold extending methods with radial control algorithm. It first does projection with arc-length along the manifold extending direction, then it control manifold extending speed by projection arc-length. The algorithm not only reaches the same result as geodesic approach, but also avoids the boundary value computation and ensures the computing efficiency.

Keywords: Manifold computing, Radial control, Interpolation method, Lorenz system, Projection.

1. Introduction

Computing two dimension manifold in three dimension space is a hard work [1]. One reason is that different trajectories extending in variable directions with different speed; another reason is because even the same trajectory changes the extending speed by time [2-3]. If the manifold is computed by constant time interval, the result of global manifold can’t reflect real dynamic feature of the system. If the manifold is computed using equal arc-length extending method, it is hard to find proper interpolation location and the errors increase by time. Specially, the errors will increase sharply when curvature of trajectories change obviously. Some corrective approaches about manifold computing have do improvements for the condition. Mark E. Johnson presented a stable flow extending approach in 1997. Generally there is some angle between the flow extending direction and the boundary of local space [4], so this method can’t ensure real stable extending. Bernd Krauskopf and Hinke Osinga [5, 6] show the geodesic approach in 1998, and the method is even appreciated now. Although the method does very well in manifold extending by computing series of boundary values instead of adjusting vector field, the boundary values computing costs too much time. Especially, when the flow extending direction is tangent with the vector field, the time cost situation become even worse. This approach constructs a pseudo manifold which is not consist of the true flows of the dynamic system [7]. Jia Meng introduced a self-adaptive parameter and trajectories continuation method in 2010 [8] and a fast computing method in 2011 [9]. Through it can get real manifold while solve time cost problem in a short time, as time increase it would fail [10-12].

2. Mathematical Setting

Suppose $F: R^n \rightarrow R^n$ is an orientation preserving diffeomorphism.
\[ x_{n+1} = F(x_n) \]  \hspace{1cm} (1)

If \( x_0 \) is a starting point, then

\[ F^{k+l}(x_0) = F^k(F^l(x_0)) \]  \hspace{1cm} (2)

\( k \) and \( l \) are integers. If \( x_0 \in \mathbb{R}^n \) meets the condition \( F(x_0) = x_0 \), then \( x_0 \) is a fixed point of \( F \). From Eq. (2) we know \( x_0 \) is also a fixed point of \( F^k \). Consider the Jacobian matrix \( A = DF(x_0) = \left[ \frac{\partial F_i}{\partial x_j} \right](x_0) \) of \( F \) at \( x_0 \). \( x_0 \) is hyperbolic if the modulus of eigenvalues of \( A \) is different from 1. The eigenvalues whose modulus is smaller than 1 are called stable and their corresponding eigenvectors \( \{v_1, v_2, \ldots, v_l\} \) span the stable eigenspace \( E^s \). The other eigenvalues whose modulus is bigger than 1 are called unstable and their corresponding eigenvectors \( \{v_{l+1}, v_{l+2}, \ldots, v_r\} \) span the unstable eigenspace \( E^u \).

**Theorem 1.** Suppose \( F: \mathbb{R}^n \to \mathbb{R}^n \) is an orientation preserving diffeomorphism and \( x_0 \) is a fixed point of \( F \), then in the neighborhood \( U \) of \( x_0 \), there exists local stable and unstable manifolds

\[ W^s(x_0) = \{x \in U : \lim_{k \to -\infty} F^k(x) \to x_0 \} \]  \hspace{1cm} (3)

\[ W^u(x_0) = \{x \in U : \lim_{k \to +\infty} F^k(x) \to x_0 \} \]  \hspace{1cm} (4)

where \( E^s \) and \( E^u \) are the tangent to \( W^s(x_0) \) and \( W^u(x_0) \) at \( x_0 \) respectively.

Global stable and unstable manifold are defined as

\[ W^s(x_0) = \{x \in \mathbb{R}^n : \lim_{k \to -\infty} F^k(x) \to x_0 \} = \bigcup_{i=0}^{\infty} F^i(W^s(x_0)) \]  \hspace{1cm} (5)

\[ W^u(x_0) = \{x \in \mathbb{R}^n : \lim_{k \to +\infty} F^k(x) \to x_0 \} = \bigcup_{i=0}^{\infty} F^{-i}(W^u(x_0)) \]  \hspace{1cm} (6)

It’s clear that the global manifold is the image of the local manifold.

Stable and unstable manifolds play an important role in analyzing the dynamics of a given system. They form basins of attraction between different attractors, and complexed dynamics like chaos, homoclinic and heteroclinic would occur when they intersect. Computation of stable and unstable manifolds will contribute to the further study of all these fields.

Several algorithms have come up. The algorithms use the idea of growing the unstable manifold and one point is found and added at a prespecified distance away from a given point each step. They have to search along the known segment on the unstable manifold back and forth to find the preimage of the new point, and it is the choke point which reduces the efficiency of the algorithm. It is verified in this paper that the gradient of the unstable manifold can be predicted by the known points on the unstable manifold and the scheme can be used to locate the preimage of the new point quickly.

According to the definition (6) of global unstable manifold, we know if \( x \in W^u(x_0) \), then \( F(x) \in W^u(x_0) \). Expand \( F \) at \( x \) as Taylor series

\[ F(x + \varepsilon_i) = F(x) + F'(x)\varepsilon_i + o(\| \varepsilon_i \|) \]  \hspace{1cm} (7)

Ignore the high order terms

\[ F(x + \varepsilon_i) = F(x) + F'(x)\varepsilon_i = F(x) + A\varepsilon_i \]  \hspace{1cm} (8)

\( A = DF(x) = \left[ \frac{\partial F_i}{\partial x_j} \right] (x) \) is the Jacobian matrix of \( F \) at \( x \).

As shown in Fig. 1, for 2D phase space

\[ \varepsilon_i = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \]  \hspace{1cm} (9)

Then

\[ E = F(x + \varepsilon_i) - F(x) = A\varepsilon_i \]  \hspace{1cm} (10)

When \( \varepsilon_i \to 0 \), it follows \( x + \varepsilon_i \to x \) and \( F(x + \varepsilon_i) \to F(x) \). So \( \Delta x / \Delta y \) can be taken as the gradient of the unstable manifold at \( x \) while \( \Delta x/\Delta y \) is the gradient at \( F(x) \).

During the computation, \( x \), \( x + \varepsilon_i \), and \( F(x) \) are known points on the unstable manifold and \( F(x + \varepsilon_i) \) is unknown. The unknown gradient of \( W^u(x_0) \) at \( F(x) \) can be approximated by known points \( x \) and \( x + \varepsilon_i \). And that is why we call our algorithm gradient prediction.
3. The Principle of Radial Control Algorithm

The radial control algorithm mainly focuses on two dimensional manifold computing in three dimension space. The general principle of the algorithm is that the trajectories extend in the radial direction with certain step length. It first does projection with arc-length along the manifold extending direction, then it control manifold extending speed by projection arc-length. At last it adjusts the step length to ensure each trajectory has the same projection arc-length. In this way, the computed manifold consists of real flows and it reflects the characteristic of global manifold well. The principle can be seen in Fig. 2.

Fig. 2. Principle of radial growth approach.

Assuming that the two dimension invariant manifold is extending at some time as Fig. 2, and the increasing factor is $\mu = C_0 / |d|$. Set the coordinate of equilibrium points $x_0$ to be $(ox, oy, oz)$, the vector of two dimensional stable subspace is $\vec{v}_1$ and $\vec{v}_2$, $P$ is the extending point of the manifold with coordinate $[x(t_i), y(t_i), z(t_i)]$, then the tangent vector at $P$ can be gotten as:

$$\vec{t} = f(x(t_i), y(t_i), z(t_i))$$  \hspace{1cm} (11)

The normal vector of eigen subspace is $\vec{n} = \vec{v}_1 \times \vec{v}_2$  \hspace{1cm} (12)

The radial vector at $P$ is $\vec{r} = P - x_0$  \hspace{1cm} (13)

Vector $\vec{n}$ and vector $\vec{r}$ decide the plane $\alpha$, and the projection of tangent vector $\vec{t}$ in $\alpha$ is

$$\vec{d} = \vec{t} - \vec{b}, \quad \vec{b} = \frac{\vec{t} \times (\vec{n} \times \vec{r})}{|\vec{n} \times \vec{r}|} = \frac{\vec{t} \times (\vec{n} \times \vec{r})}{|\vec{n} \times \vec{r}|}$$  \hspace{1cm} (14)

With all equations above, the increase factor can be show as:

$$\mu = \frac{C_0 |\vec{n} \times \vec{r}|}{||\vec{n} \times \vec{r}||^2 \vec{t} - \vec{t} \times (\vec{n} \times \vec{r})(\vec{n} \times \vec{r})}$$  \hspace{1cm} (15)

For $x_0$, $P$, $\vec{v}_1$, $\vec{v}_2$, $f$, $\vec{n}$, $\vec{r}$, $\vec{t}$ are known quantity, and $C_0$ is a constant that has been set before, then the extending speed normalization in radial direction of the dynamic system is

$$\frac{dx}{dt} = \mu f(x)$$  \hspace{1cm} (16)

Extending speed normalization in radial direction is the equivalent transformation, which ensures the flows of the original and transformed dynamic systems keep invariant.

Computing steps:

Step 1. Through the Newton - posen algorithm system singular value point, and based on the Jacobi matrix of at this point and two-dimensional stable or unstable feature subspace, as the base domain of manifolds.

Step 2. Select the base domain boundary point as the starting point of the global manifolds.

Step 3. Interpolation and delete by selecting rule of interpolation points, to prepare for the formulation of the grid.

Step 4. The preparation of grid function called grid.

4. The Simulation Results Comparison

Taking the Lorenz model as an example, we could do the simulation. The equations of Lorenz were described as:

$$\begin{cases} \dot{x} = -\delta(x - y) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = -\beta z + xy \end{cases}$$  \hspace{1cm} (17)

When $\sigma = 10$, $\rho = 28$, and $\beta = 8/3$, the attractor is chaotic, as shown in Fig. 3(a). The model is continuous and in the form of an ordinary differential equation. By using difference, the system is described as:
\[
\begin{align*}
\frac{x_{n+1} - x_n}{T} &= \sigma (y_n - x_n) \\
\frac{y_{n+1} - y_n}{T} &= x_n (\rho - z_n) - y_n \\
\frac{z_{n+1} - z_n}{T} &= x_n y_n - \beta z_n
\end{align*}
\] (18)

The previous is simplified as
\[
\begin{align*}
x_{n+1} &= T \sigma (y_n - x_n) + x_n \\
y_{n+1} &= T x_n (\rho - z_n) - T y_n + y_n \\
z_{n+1} &= T (x_n y_n - \beta z_n) + z_n
\end{align*}
\] (19)

In order to maintain the property of the continuous Lorenz system, the value of \( T \) need to be appropriate. If \( T \) is too great, the approximation is too coarse and the discrete system is not chaotic anymore; on the other hand, if \( T \) is too small, the evolution speed of the system is too slow. We find that when \( T = 0.01 \), the discrete Lorenz system has a chaotic attractor (shown in Fig. 3(b)) similar to that of the continuous Lorenz system, at the same time, the system evolves at a moderate speed.

The origin is a hyperbolic fixed point of the discrete Lorenz system, and the Jacobian matrix at it is
\[
A = \begin{bmatrix}
0.9 & 0.1 & 0 \\
0.28 & 0.99 & 0 \\
0 & 0 & 0.9733
\end{bmatrix}
\]

Jacobian matrix \( A \) has 3 real eigenvalues: \( \lambda_1 = 0.7717 \), \( \lambda_2 = 1.1183 \) and \( \lambda_3 = 0.9733 \). It is interesting to notice that the discrete Lorenz has 2D stable manifold, which is also similar to that of the continuous Lorenz system.

From Fig. 4 to Fig. 7, we can see that the results of radial control approach as Fig. 4 and geodesic approach as Fig. 6 looked the same. In the two figures, each trajectory keeps the same increasing step length in radial direction. Through the results of Fig. 4 and Fig. 6 looked alike, the principle is different. The radial control approach are real flow extending, all points are the true value of the manifold. It only has errors coming from iteration. The geodesic approach does some interpolation, so its error consists of two parts. One part from interpolation and another from iteration. When removing the mesh and keep the flow, Fig. 4 will become Fig. 7. The result of normalization method as in Fig. 4 looked worse. The extending speed in different directions is variable, while the curvature of trajectories keeps change.
5. Conclusions

In this paper, the radial control method in the aspect of equilibrium manifold growth in all directions to show the good performance, its effect and even to be able to achieve the requirement of the geodesic method. Compared with the geodesic method, the radial control method at the expense of the disadvantages of tangential velocity to keep the cost of the radial velocity, is not conducive to control the size of the tangential velocity and change; Advantage is that can be drawn in the form of discrete flow manifold, so as to better reaction flow and the direction of the change rule, and can be combined with the existing algorithms for the calculation of manifold, simple calculation and widely used. It is important to note that after the normalized radial dynamical system, the independent variable is no longer the time, but the arc length of a constant time. In the example in this paper, referred to in a certain time interval on the calculation of the manifold, in fact, they are no longer the original time scale, but the independent variable of unified appellation, distinguish between. The disadvantage of radial control algorithm is that it sacrifices the tangential direction extending speed to keep radial direction extending speed. It can not reflect the changes alone tangential direction. The radial control algorithm still can’t solve the problem that when the tangent vector is tangent is with the boundary of local manifold, the value of $\mu$ in equation trend to be infinite. In this case, there must be limit cycle, which occur in the same plane as the local subspace.

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