Bifurcation in Ground-state Fidelity and Quantum Criticality in Two-leg Potts Ladder

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Received: 7 January 2014 / Accepted: 7 February 2014 / Published: 28 February 2014

Abstract: We have investigated an intriguing connection between bifurcations, reduced fidelity per lattice site, local order parameter, universal order parameter, entropy and quantum phase transitions in the ground state for quantum three-state Potts model with two coupled infinite-size ladder system, in the context of the tensor network algorithm. The tensor network algorithm produces degenerate symmetry-breaking ground-state wave functions arising from the Z3 symmetry breaking, each of results from a randomly chosen initial state. We expect that our approach might provide further insights into critical phenomena in quantum many-body infinite lattice systems in condensed matter physics. Copyright © 2014 IFSA Publishing, S. L.

Keywords: Bifurcation, Reduced fidelity, Order parameter, Entropy, Phase transition, Tensor network algorithm.

1. Introduction

Quantum many body systems are very hard to study due to the exponential growth of their Hilbert space with the number of constituents. The interest in quantum phase transitions (QPTs) in magnetic systems, which can be triggered by varying some of the system’s parameters, is increased. QPTs occurred in strongly correlated systems at low temperatures and driven by the quantum fluctuations and the competition between interactions.

In the past few decades, the study of spin ladders attracted the subject of extensive experimental and theoretical interest [1] with the development of the numerical simulation. One of the most striking feature of the spin-1/2 Heisenberg ladders is that their spin excitations spectra are gapful (gapless) when the numbers of legs is even (odd) [2]. Besides, the spin ladders represent particularly interesting class of models often exhibiting some attractive rich variety of phases [3-5]. Recently, considerable attention has turned to two-leg quantum spin-ladder systems. The studies of the two-leg spin ladder had been investigated by various methods [5-10].

On the other hand, significant progress has been made in developing efficient numerical algorithms to simulate quantum many-body lattice systems in the context of the tensor network (TN) representations. By computing the ground-state fidelity per lattice site for our understanding of quantum entanglement and fidelity, these algorithms characterize critical phenomena in a variety of quantum many-body
lattice systems in any spatial dimensions [11-17]. The study of quantum condensed matter systems is benefiting from an infusion of ideas related to quantum information and entanglement. Fidelity as a measure of quantum state distinguishability in quantum information science, describes the distance between two given quantum states, which is a novel approach to quantum phase transitions. In Refs. [11-13], it was argued the ground-state fidelity per lattice site is able to characterize changes of the ground-state wave functions around a critical point. Many powerful numerical algorithms have been developed in the context of the TN representation, an adaptation of the tensor network representation algorithm for ladder is investigate two-leg three-state Potts ladder.

For a QPT arising from a spontaneous symmetry breaking (SSB), a bifurcation appears in the ground-state fidelity per site, with a bifurcation point identified as a phase transition point. Similarly, a bifurcation occurs in the ground-state reduced fidelity between the one-site reduced density matrices and the two-site reduced density matrices, with a bifurcation point identified as a phase transition point for quantum many-body spin ladder lattice systems. A systematic scheme to study critical phenomena in quantum many-body lattice systems is presented by computing the ground-state fidelity per lattice site, deriving local order parameters (if any) from the reduced density matrices for a representative ground-state wave function in a given phase, and characterizing any phase without any long range order. And the universal order parameter [18] appears as the ground-state fidelity per lattice site between a ground state and its symmetry-transformed counterpart.

Quantum entanglement plays a key role near the quantum critical point in quantum spin-ladder systems. The ground state is highly entangled, and the QPT should be characterized by the abrupt change of the quantum entanglement [19-21], can be probed by the entropy for quantum spin-ladder systems. In this paper, firstly, we introduce the infinite-size two-leg three-state Potts ladder model. Then, we investigate the ground-state fidelity per lattice site for infinite-size two-leg three-state Potts ladder by using the efficient TN algorithm which enables us to efficiently generate ground-state wave functions [22]. Next, with the bifurcations in the ground-state fidelity per lattice site and the bifurcations occurring in the ground-state reduced fidelity between the one-site, two-site and four-site reduced density matrices, we successfully detect the phase transitions and capture drastic changes of ground-state wave functions around critical points.

Finally, the physical observable parameters, the local order parameter, universal order parameter, and the entropy for quantum spin-ladder systems are given. The phase transition points detected by the above parameters are consistent with the bifurcation results.

2. Model

We consider the quantum two-leg three-state Potts model in a transverse magnetic field on an infinite-size ladder. The Hamiltonian takes the form

\[ H = H_{\text{leg}} + H_{\text{rung}}, \]

where

\[ H_{\text{leg}} = -\sum_{i} \sum_{\alpha=1,2} (M_{i1,\alpha}^i M_{i2,\alpha}^{i+1} + M_{i2,\alpha}^i M_{i1,\alpha}^{i+1} + h M_{i,\alpha}^\beta); \]

\[ H_{\text{rung}} = -\sum_{i} (M_{i1,1}^i M_{i2,2}^i + M_{i2,1}^i M_{i1,2}^i + h M_{i,2}^\beta) \]

\( h \) is the transverse magnetic field; \( M_{i,\beta,a}^\beta (\beta = 1,2) \) are the Potts spin matrices at site \( i \) on the \( \alpha \)-th leg:

\[
M_{i1,\alpha} = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix},
M_{i2,\alpha} = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

and

\[
M_{i} = \begin{pmatrix}
2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix},
\]

The model is invariant with respect to the operations \( M_{i1,\alpha}^i \rightarrow \omega M_{i1,\alpha}^i \), \( M_{i2,\alpha}^i \rightarrow \omega^2 M_{i2,\alpha}^i \), and \( M_{i,\beta,a}^\beta \rightarrow \omega^\beta M_{i,\beta,a}^\beta \), for all the sites simultaneously, which yields the \( \mathbb{Z}_3 \) symmetry. The model is expected to undergo a continuous QPT [23]. Here, in terms of the newly-developed TN algorithm [22], we obtain the ground state wave function, focus on the computation of the ground-state fidelity per lattice site, reduced fidelity per lattice site, local order parameter, super order parameter and entropy to detect the phase transitions of systems.

3. The Ground-State Fidelity Surface

The ground-state wave function is computed in terms of the newly-developed TN algorithm for spin ladders [22]. We investigate the ground-state fidelity per lattice site to demonstrate that the phase transitions of the system characterized by some pinch points on the fidelity surface marking some phase transitions, which is accepted as a universal marker to...
identify quantum criticalities: a phase transition point is characterized by a pinch point on the fidelity surface.

For the two-leg ladder with \( h \) as the control parameter, with two ground-state wave functions \( |\psi(h_1)\rangle \) and \( |\psi(h_2)\rangle \) corresponding to two different values \( h_1 \) and \( h_2 \). The ground-state fidelity \( F(h_1, h_2) = \langle \psi(h_2) | \psi(h_1) \rangle \) asymptotically scales as \( F(h_1, h_2) - d(h_1, h_2)^2 \), with \( N \) being the number of sites in the lattice. Here, \( d(h_1, h_2) \) is the scaling parameter, introduced for one-dimensional quantum systems and for two and higher-dimensional quantum systems, which characterizes how fast fidelity goes to zero when the thermodynamic limit is approached. Physically, the scaling parameter \( d(h_1, h_2) \) is the averaged fidelity per lattice site, which is well defined in the thermodynamic limit:

\[
\text{Ind}(h_1, h_2) \equiv \lim_{N \to \infty} \frac{F(h_1, h_2)}{N}. \tag{4}
\]

It satisfies the properties inherited from the fidelity \( d(h_1, h_2) \) : (i) normalization \( d(h, h) = 1 \); (ii) symmetry \( d(h_1, h_2) = d(h_2, h_1) \); and (iii) range \( 0 \leq d(h_1, h_2) \leq 1 \).

The TN representation allows to efficiently computing the ground-state fidelity per lattice. By computing the ground-state fidelity per lattice site, we are able to characterize different phases by some pinch points.

In Fig. 1, we plot the ground-state fidelity per site, \( d(h_1, h_2) \), for the quantum three-state Potts model in a transverse magnetic field on an infinite-size two-leg ladder as a function of \( h_1 \) and \( h_2 \).

![Fig. 1. A two-dimensional fidelity surface embedded in a three-dimensional Euclidean space for the quantum three-state Potts model in a transverse magnetic field on an infinite-size two-leg ladder, with the transverse magnetic field \( h \) as the control parameter with truncation dimension \( D=6 \).](image)

A continuous phase transition point \( h_c = 1.77 \) is identified as a pinch point \((h_c, h_c)\) on the fidelity surface. The critical point \( h_c = 1.77 \) occurs as a pinch point on the ground-state fidelity surface.

Here, a pinch point, as expected, is seen in the ground-state fidelity surface around the exact critical point \( h_c = 1.77 \). Our result of the pinch point on a fidelity surface for a quantum many-body lattice system undergoing a continuous QPT, is in a good agreement with the previous studies [23].

### 4. Bifurcation in the Ground-state Fidelity Per Lattice Site

A bifurcation occurs in the ground-state fidelity per lattice site [11], \( d(h_1, h_2) \), as a function of \( h_1 \), for a fixed \( h_2 \). If \( |\psi(h_2)\rangle \) is a reference state, with \( h_1 \) in the symmetric phase, then the ground-state fidelity per lattice site, \( d(h_1, h_2) \), cannot distinguish degenerate ground states in the symmetry-broken phase. However, if \( |\psi(h_2)\rangle \) is a reference state, with \( h_2 \) in the symmetry-broken phase, then the ground-state fidelity per lattice site, \( d(h_1, h_2) \), can be used to distinguish degenerate ground states in the symmetry-broken phase. So, a phase transition point \( h_c \) manifests itself as a bifurcation point [24] with truncation dimension \( D=6 \).

In Fig. 2, we plot the ground-state fidelity per lattice site, \( d(h_1, h_2) \), for the quantum two-leg three-state Potts ladder, with the transverse magnetic field \( h \) as the control parameter. Here, we choose \( |\psi(h_2)\rangle \) as a reference state, with \( h_2 \) in the Z3 symmetry-broken phase, then \( d(h_1, h_2) \) is able to distinguish three degenerate ground states, with a phase transition point \( h_c \) as bifurcation points. A phase transition point at \( h_c = 1.77 \) is identified from the bifurcation in the ground-state fidelity per lattice site for the truncation dimension \( D=6 \).

An intriguing feature of the bifurcation points for the ground-state fidelity per lattice site, as seen in Fig. 2, is that the ground-state fidelity per lattice site between different symmetry breaking ground states in the same phase is always lower than that between ground states from different phases. However, this is not unexpected, this simply means that degenerate symmetry breaking ground states in the same symmetry-broken phase are more distinguishable than ground states from different phases.

If we choose \( |\psi(h_2)\rangle \) as a reference state, with \( h_2 \) being in the Z3 symmetry-broken phase, then \( d(h_1, h_2) \) distinguishes three degenerate ground-state wave functions, with a phase transition point \( h_c \) as a bifurcation point. Here, we have chosen \( h_2 = 0.5 \). The
phase transition point is identified at $h_c=1.77$ for the truncation dimension $D=6$.

Fig. 2. The ground-state fidelity per lattice site, $d(h, h_2)$, for the quantum three-state Potts model in a transverse magnetic field on an infinite-size two-leg ladder, with the transverse magnetic field $h$ as the control parameter.

5. Reduced Fidelity Per Lattice Site

For quantum two-leg spin ladder, the ground-state reduced fidelity $F(h_1, h_2)$ is defined as [25]

$$F(h_1, h_2) = \text{Tr} \left[ \sqrt[2]{\rho_{h_1}^2 \rho_{h_2}^2} \right], \quad (5)$$

where $\rho_{h_1}$ and $\rho_{h_2}$ are the reduced density matrices corresponding to two different control parameter values, $h_1$ and $h_2$.

In Fig. 3, the plot shown the ground-state one-site reduced fidelity $F_1$, two-site reduced fidelity along the leg direction $F_{2\text{-leg}}$, two-site reduced fidelity along the rung direction $F_{2\text{-rung}}$, four-site reduced fidelity $F_3$ as a function of $h$ for quantum two-leg for the quantum three-state Potts ladder, with $\rho_{h_1}$ and $\rho_{h_2}$ the one-site, two-site and four-site reduced density matrices corresponding to $h_1$ and $h_2$, respectively. Here, we choose $\rho_{h_2}$, with $h_2=0.5$, as a reference state, which breaks the $Z_3$ symmetry. The phase transition point locates at $h_c=1.77$ for the truncation dimension $D=6$. When the control parameter $h$ crosses a transition point, the ground-state degeneracy changes suddenly, implying that the system undergoes a QPT. Here, the critical point $h_c$ is shown no significant shift with the truncation dimension $D$ increasing.

For the ground-state one-site reduced fidelity $F_1$, two-site reduced fidelity along the leg direction $F_{2\text{-leg}}$, two-site reduced fidelity along the rung direction $F_{2\text{-rung}}$, four-site reduced fidelity $F_3$, we have chosen $\rho_{h_2}$ as the reference state, with $h_2=0.5$ in the $Z_3$ symmetry-broken phase. In all cases, the phase transition point locates at $h_c=1.77$ for the truncation dimension $D=6$.

Fig. 3. The ground-state reduced fidelity, $F(h_1, h_2)$, for quantum two-leg three-state Potts model in a transverse magnetic field on an infinite-size ladder.

6. Local Order Parameter

A central concept in the conventional Landau-Ginzburg-Wilson paradigm is local order parameters, whose nonzero values characterize a given phase. The local order parameter for quantum model is not universal and even not obvious. However, once the ground-state phase diagram is known, we are able to extract the local order parameter from the reduced density matrices for a representative ground-state wave function in a given phase.

For quantum three-state Potts model in a transverse magnetic field on an infinite-size two-leg ladder, it turns out that the one-site reduced density matrix $\rho$ displays different nonzero-entry structures in two phases. The spontaneous magnetization of quantum three-state Potts model in a transverse magnetic field on an infinite-size two-leg ladder, which is defined as,

$$O_p = |\langle M_{x,1} + M_{x,2} \rangle|/2.$$  \hspace{1cm} (6)

The spontaneous magnetization $O_p$ exhibits a long-range order. The system has the spontaneous magnetization for $h<1.77$ (Fig. 4).

Upper panel: The spontaneous magnetization, which is defined as $|\langle M_{x,1} + M_{x,2} \rangle|/2$, as a function of the transverse magnetic field strength $h$ for quantum two-leg three-state Potts model in a transverse magnetic field on an infinite-size ladder.

Lower panel: The universal order parameter $I(h)$ for quantum two-leg three-state Potts model in a transverse magnetic field on an infinite-size ladder. A phase transition point $h_c$ occur, as $I(h)$ changes from being nonzero to zero at $h_c=1.77$. Here, the bond truncation dimension is $D=6$. 

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7. Universal Order Parameter

For any quantum lattice system with a symmetry group \( G \) undergoing a QPT with symmetry breaking order, as argued in Ref. [18], a universal order parameter appears, which is discontinuous for first-order phase transitions and continuous for second-order phase transitions. For any ground state \( |\psi\rangle \) in the symmetric phase, 
\[
||g|\psi\rangle\langle\psi|| = 1
\]
for any symmetry operation \( g \in G \), whereas it is identical to zero in the symmetry broken phase.

As argued in Ref. [11], for a finite size system with \( N \) particles, it is found that the distance between two quantum states \( |\psi\rangle_N \) is written as
\[
||g|\psi\rangle_N\langle\psi|| \approx f^N(h)
\]
where \( f^N(h) \) is the averaged fidelity per lattice site, which is well-defined even in the thermodynamic limit. Then, \( f(h) = 1 \) for any \( g \in G \), if \( h \) is in the symmetric phase, and \( 0 < f(h) < 1 \) for any nontrivial symmetry operation \( g \), if \( h \) is in the symmetry-broken phase. As argued in Ref. [18], the universal order parameter as follows,
\[
I(h) = \sqrt{1 - f^2(h)}.
\]

Note that \( I(h) \) is always zero if \( h \) is in the symmetric phase. However, it becomes nonzero, with its value ranging from 0 to 1, if \( h \) is in the symmetry-broken phase. These features are exactly meet requires for \( I(h) \) to be an order parameter. In fact, this is valid for any quantum many-body lattice system with a global symmetry group \( G \) spontaneously broken.

In Fig. 4 (lower panel), we plot the universal order parameter \( I(h) \) for quantum two-leg three-state Potts model on an infinite-size ladder, with the transverse magnetic field \( h \) as the control parameter. Here, the symmetry operation is the non-trivial element \( g \) of the group \( Z3 \). If \( h < h_c \), the universal order parameter \( I(h) \) is non-zero. This characterizes the Z3 symmetry phase, in contrast to the fact that the universal order parameter \( I(h) \) is zero. When the control parameter \( h \) varies across the critical point \( h_c \), the behavior of the universal order parameter \( I(h) \) changes qualitatively, implying that the system undergoes a QPT at the phase transition point \( h_c = 1.77 \). As the truncation dimension \( D \) is increased, the phase transition point \( h_c \) no obvious shift.

8. Entropy

In the past years it has been recognized that quantum entanglement is the key concept to understand the intricate structure of many body wave functions in condensed matter physics.

Quantum entanglement has close relationship with QPTs in many-body systems. In this section, we will briefly evaluate the entanglement measures. The TN with translational invariance can also be seen as the entanglement measure of the ground states. The entropy is a measure of an entanglement present in a quantum state, whose behavior is universal in many occasions. The entropy, which tells us how disorder of the system, of a pure state for a system partitioned into two parts \( A \) and \( B \) is defined as
\[
S = -Tr\rho_A \log \rho_A = -Tr\rho_B \log \rho_B,
\]
where \( \rho_A(\rho_B) \) is the reduced density matrix of the subsystem \( A(B) \) for a given matrix dimension \( D \). So, we can straight forwardly take the form
\[
S = -Tr\rho_i \log \rho_i
\]
where \( \rho_i \) is the ground-state reduced density matrix of one- or multi-site. It is thus obvious that the TN representations with matrices of finite size cannot describe exactly the behavior of an infinite system at the critical point but we may try to find the exact amount of entanglement which is captured.

Here, the single-site entropy, which is denoted as \( S1 \), with \( \rho_i \) is the ground-state one-site reduced density matrix. For comparison, we will also evaluate the two-site and four-site entropy, which is denoted as \( S2\text{-}\text{leg}, S2\text{-}\text{rung}, \) and \( S3 \), with \( \rho_i \) are the ground-state two-site reduced density matrix along the leg.
direction, the ground-state two-site reduced density matrix along the rung direction, and four-site reduced density matrix, respectively.

Our results shown that the entanglement measures are equally good for characterizing the quantum criticality by their derivative discontinuities right at the critical point for quantum two-leg three-state Potts model with two coupled infinite-size spin ladder system. It shows that the entanglement measures are also capable of characterizing the quantum phase transitions for quantum infinite-size spin ladder system.

In Fig. 5, the entropy S1, S2-leg, S2-rung, and S3 are plotted for quantum three-state Potts model in a transverse magnetic field on an infinite-size two-leg ladder. The entanglement measures per lattice site S1, the entanglement measures per two-lattice-site along the leg direction S2-leg, the entanglement measures per two-lattice-site along the rung direction S2-rung, and the entanglement measures per four-lattice-site S3, for quantum two-leg three-state Potts model on an infinite-size ladder, with transverse magnetic field $h$ as the control parameter, respectively. The derivative of these quantities shows a discontinuity at $h_c = 1.77$. Here, the bond truncation dimension is $D=6$.

![Fig. 5. The entropy S1, S2-leg, S2-rung, and S3 for quantum three-state Potts model in a transverse magnetic field on an infinite-size two-leg ladder.](image)

The entanglement measures per lattice site S1, the entanglement measures per two-lattice-site along the leg direction S2-leg, the entanglement measures per two-lattice-site along the rung direction S2-rung, and the entanglement measures per four-lattice-site S3, for quantum two-leg three-state Potts model on an infinite-size ladder, with transverse magnetic field $h$ as the control parameter. The derivative of these quantities shows a discontinuity at $h_c = 1.77$. Here, the bond truncation dimension is $D=6$.

9. Conclusions

We have investigated the behaviors of bifurcations of both fidelity per lattice-site and reduced fidelity per lattice-site, local order parameter, universal order parameter and entropy, and demonstrated an intriguing connection between those physical observables and QPTs in the ground state for quantum two-leg three-state Potts model with two coupled infinite-size spin ladder system, in the context of the TN algorithm. For quantum two-leg three-state Potts model with two coupled infinite-size spin ladder system, the TN algorithm produces degenerate symmetry-breaking ground-state wave functions arising from the Z3 symmetry breaking, each of the results from a randomly chosen initial state. Therefore, a quantum system undergoing a phase transition is characterized in terms of SSB that is captured by a bifurcation in the ground-state fidelity per lattice site.

Acknowledgements

This work is supported in part by the National Natural Science Foundation of China (Grant No: 11104362), and was supported by Chongqing City Board of Education Science and technology research projects.

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