Multisensor Weighted Measurement Fusion Kalman Filter with Correlated Noises System

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Abstract: For the multisensor stochastic control systems with different measurement matrices and correlated noises, a new weighted measurement fusion (WMF) estimation algorithm is presented by using full-rank decomposition of matrix and weighted least squares theory. The newly presented algorithm can handle the fused filtering, smoothing, and prediction problems for the state in a unified framework and prove the global optimality, which indicates that the estimating result is equivalent to centralized fusion (CF) Kalman estimating result. However, it can obviously reduce the computational burden compared with CF, so it is convenient for application in real time. A simulation result will show the effectiveness of the proposed algorithm.

Keywords: Correlated noises, Weighted measurement fusion, Full-rank decomposition, Centralized fusion.

1. Introduction

Recently, weighted measurement fusion method has gained great attention, in that it can not only reduce computational burden greatly, but also can give global optimal estimation precision, which can be explained as its estimation precision and centralized fusion filter estimation precision are the same [1-3], so it is globally optimal.

Its basic principles are that it firstly converts the measurements of many sensors into an equivalent sensor, according to some fusion rules, then the equivalent sensor system is filtered, so weighted measurement fusion method is also called data compression fusion or combined measurement fusion, but it is considered short of flexibility in [3-7]. Q. Gan and C. J. Harris put forward the weighted measurement fusion algorithm with the assumption that all the sensors have the same measurement matrix and the measurement noises of each sensor are not correlated [4]. Lagrange multiplier method presented in [5] has solved the constraint that the measurement noises of each sensor are not correlated required in [4], but the constraint that all the sensors must have the same measurement matrix is not solved. To overcome the constraint, improvements are made in [6] and [7], while requiring the column rank of measurement matrix of the extended matrix of all the sensors full or the measurement matrix of all the sensors having the maximal right factor. Self-tuning weighted measurement fusion Kalman filtering algorithm is presented in [8-11].

This paper presents a new weighted measurement fusion algorithm using matrix full-rank decomposition and weighted least squares theory, and proves its global optimality, under the conditions of each sensor node’s measurement matrix different.
from each, another sensor’s measurement noise correlated, input noises and measurement noises correlated. Meanwhile, the method of ordinary decomposition is used to prove the method presented in [3-7] is just the special case of the algorithm presented in this paper.

2. Problem Formulation

Consider the following discrete time-invariant multisensor system with correlated noises and different measurement matrix

\[ x(t + 1) = \Phi x(t) + Bu(t) + \Gamma w(t), \]

\[ y_i(t) = H_i x(t) + v_i(t) + b_i, \quad i = 1, \ldots, L, \]

where \( t \) is the discrete time, \( x(t) \in \mathbb{R}^n \) is the state, \( u(t) \in \mathbb{R}^m \) is the given control, \( y_i(t) \in \mathbb{R}^n_i \), \( i = 1, 2, \ldots, L \), is the measurement of the \( i \)th sensor, \( b_i \) is the system deviation, \( H_i \in \mathbb{R}^{n_l \times n} \) is the measurement matrix of the \( i \)th sensor, \( w(t) \) and \( v_i(t) \) are the correlated white noise with zero means, i.e.

\[
E\left[ \begin{bmatrix} w(t) \\ v_i(t) \end{bmatrix} [w^T(k) \ v_j^T(k)] \right] = \begin{bmatrix} Q_i & S_i \\ S_i^T & R_i \end{bmatrix} \delta_{ij},
\]

where \( \delta_{ii} = 1, \delta_{ij} = 0 (i \neq j) \). \( Q_i \) is the variance of \( w(t) \), \( S_i \) is the correlated matrix of \( w(t) \) and \( v_i(t) \). The cross-covariance of \( v_i(t) \) and \( v_j(t) \) is \( R_{ii} = R_{jj}^T, (i, j = 1, \ldots, L, i \neq j) \). Suppose \( R_0 = (R_0)_{LL} > 0 \), a new measurement signal of \( z(t) = y_i(t) + b_i \) is introduced and \( L \) measurement equations of (2) are integrated yielding

\[ y^{(i)}(t) = H^{(i)} x(t) + v^{(i)}(t), \]

where

\[ y^{(i)}(t) = [z_1^{(i)}(t), \ldots, z_L^{(i)}(t)]^T, \]

\[ H^{(i)} = [H_1^{(i)}, \ldots, H_L^{(i)}]^T, \]

\[ v^{(i)}(t) = [v_1^{(i)}(t), \ldots, v_L^{(i)}(t)]^T. \]

The correlated function of \( w(t) \) and \( v^{(i)}(t), S_i = [S_1, \ldots, S_L] \). The variance matrix of \( v^{(i)}(t) \) is \( R^{(i)} \). To convert system (1) and (4) into uncorrelated system, from (4), (1) is equivalent to

\[ x(t + 1) = \Phi x(t) + Bu(t) + \Gamma w(t) + J y^{(i)}(t) - H^{(i)} x(t) + v^{(i)}(t), \]

where \( J \) is the undecided matrix. (5) can be converted into

\[ x(t + 1) = \Phi x(t) + Bu(t) + \Gamma w(t) + J y^{(i)}(t), \]

\[ \bar{\Phi} = \Phi - JH^{(i)}, \]

\[ \bar{\pi}(t) = Bu(t) + J y^{(i)}(t), \]

\[ \bar{w}(t) = \Gamma w(t) - J y^{(i)}(t), \]

where \( J y^{(i)}(t) \) as output feedback becomes a part of controlling item. Then primary system (1) and (2) are equivalent to system formed by (4) and (6).

To make \( E[\bar{w}(t) y^{(i)}(t)] = 0 \) established, introducing \( J = \Gamma SR^{(i)-1} \), which ensures that \( \bar{\pi}(t) \) and \( y^{(i)}(t) \) are not correlated white noises. Then variance matrix of \( \bar{w}(t) \) is yielded as

\[ \bar{Q}_w = \Gamma (Q_v - SR^{(i)} S^T) \Gamma^T \]

From [12], we know that any nonzero matrix \( H^{(i)} \) has full-rank decomposition

\[ H^{(i)} = FH^{(ii)}, \]

where \( F \) is the column full-rank matrix, and \( H^{(ii)} \) is the row full-rank matrix, then measurement model (4) can be represented as

\[ y^{(i)}(t) = FH^{(ii)} x(t) + v^{(i)}(t) \]

Given that \( F \) is the column full-rank matrix and from (4), we know \( F^T R^{(i)} F \) is nonsingular. Then weighted least squares (WLS) [12] method is used and Gauss-Markov estimation value of \( H^{(ii)} x(t) \) is yielded as

\[ y^{(ii)}(t) = \Omega y^{(i)}(t) \]

\[ \Omega = (F^T R^{(ii)-1} F)^{-1} F^T R^{(ii)-1} \]

substituting (10) into (11) yields

\[ y^{(ii)}(t) = H^{(ii)} x(t) + v^{(ii)}(t) \]

\[ v^{(ii)}(t) = \Omega y^{(i)}(t) \]

\[ R^{(ii)} = E[ y^{(ii)}(t) y^{(ii)T}(t)] = (F^T R^{(ii)-1} F)^{-1} \]

The standard Kalman filtering algorithm [2] is used respectively for the system of (4) and (6) and the system of (6) and (13), we can get CF and WMF Kalman estimator of \( \hat{x}^{(i)}(t | t + j) \), \( i = I, II \), and the error variances of \( P^{(i)}(t | t + j) \), \( i = I, II \). The problem is to prove whether Kalman state estimator and error variance matrix computed by these two fusion algorithms (CF and WMF) are equivalent in numeric value.
3. Global Optimality

The equations (4) and (13) can be integrated as
\[ y^{(i)}(t) = H^{(i)}x(t) + v^{(i)}(t), \quad i = I, II, \]  
(15)

Kalman filter and predicator of system (6) and (15) are respectively as following
\[ \hat{x}^{(i)}(t+1|t+1) = \hat{x}^{(i)}(t+1|t) + K^{(i)}(t)\epsilon^{(i)}(t+1|t+1), \]  
(16)
\[ \hat{x}^{(i)}(t+1|t+1) = \hat{x}^{(i)}(t+1|t) + \bar{F}^{(i)}(t|t) + \bar{O}(t), \]  
(17)
\[ \epsilon^{(i)}(t+1) = y^{(i)}(t+1) - H^{(i)}\hat{x}^{(i)}(t+1|t+1) \]  
(18)

Filtering gain matrix can be computed equivalently by the following [2]
\[ K^{(i)}(t+1) = P^{(i)}(t+1|t)H^{(i)T}(H^{(i)}P^{(i)}(t+1|t)H^{(i)T} + R^{(i)})^{-1}, \]  
(19)
\[ K^{(i)}(t+1) = P^{(i)}(t+1|t+1)H^{(i)T}R^{(i)T}, \]  
(20)

error variances matrix of one-step predictor and filtering are
\[ P^{(i)}(t+1|t) = \tilde{F}P^{(i)}(t|t)\tilde{F} + \bar{O}, \]  
(21)
\[ P^{(i)}(t+1|t+1) = [I - K^{(i)}(t+1|t+1)H^{(i)T}]P^{(i)}(t+1|t) \]  
(22)

with initial value \( \hat{x}^{(i)}(0|0), P^{(i)}(0|0), i = I, II. \)

Define \( P^{(i)-1}(t|t) = (P^{(i)}(t|t))^{-1}, \)
\[ P^{(i)-1}(t+1|t) = (P^{(i)}(t+1|t))^{-1} \]  
as information matrix, setting
\[ \hat{\alpha}^{(i)}(t | t) = P^{(i)-1}(t|t)\hat{x}^{(i)}(t|t), \quad \hat{\alpha}^{(i)}(t+1|t) = P^{(i)-1}(t+1|t)\hat{x}^{(i)}(t+1|t). \]  
We have Kalman information filter
\[ \hat{\alpha}^{(i)}(t+1|t) = P^{(i)-1}(t+1|t)\tilde{F}P^{(i)}(t|t)\hat{\alpha}^{(i)}(t|t) + P^{(i)-1}(t+1|t)\bar{O}(t), \quad i = I, II, \]  
(23)
\[ \hat{\alpha}^{(i)}(t | t) = \hat{\alpha}^{(i)}(t | t-1) + H^{(i)T}R^{(i)-1}\epsilon^{(i)}(t) \]  
(24)
\[ P^{(i)}(t+1|t) = \tilde{F}P^{(i)}(t|t)\tilde{F} + \bar{O}, \]  
(25)
\[ P^{(i)-1}(t+1|t) = P^{(i)-1}(t+1|t) + H^{(i)T}R^{(i)T}H^{(i)} \]  
(26)

Theorem 1. Multisensor system (1) and (2), weighted measurement fusion Kalman filter, one-step predicator and their error variance matrix are respectively equivalent to centralized fusion Kalman filter, predicator and their error variance matrix, i.e.
\[ \hat{x}^{(i)}(t | t) = \hat{x}^{(II)}(t | t), \quad \forall t \]  
(27)
\[ \hat{x}^{(I)}(t+1|t) = \hat{x}^{(II)}(t+1|t), \quad \forall t, \]  
(28)
\[ P^{(i)}(t | t) = P^{(II)}(t | t), \quad \forall t \]  
(29)
\[ P^{(I)}(t+1|t) = P^{(II)}(t+1|t), \quad \forall t \]  
(30)

provided that they have the same initial value
\[ \hat{x}^{(I)}(0|0) = \hat{x}^{(II)}(0|0), \]  
(31)

Proof from (9)-(14), we have
\[ H^{(I)T}R^{(I)-1} = \frac{H^{(II)T}(F^{T}R^{(II)-1}F)H^{(II)}}{H^{(I)T}R^{(I)-1} - H^{(II)T}(F^{T}R^{(II)-1}F)H^{(II)}}, \]  
(32)
\[ H^{(I)T}R^{(I)-1}y^{(I)}(t) = \frac{H^{(II)T}(F^{T}R^{(II)-1}F)(F^{T}R^{(II)-1}F)^{-1} - F^{T}R^{(II)-1}y^{(II)}(t)}{H^{(I)T}R^{(I)-1} - H^{(II)T}(F^{T}R^{(II)-1}F)H^{(II)}}, \]  
(33)
filtering error initial value \( \hat{x}^{(I)}(0|0) = x(0) - \hat{x}^{(II)}(0|0) \)
\( i = I, II, \) then from (31), we know \( \hat{x}^{(I)}(0|0) = \hat{x}^{(II)}(0|0), \) so the initial value of error variance matrix \( P^{(I)}(0) = \tilde{P}^{(II)}(0), \) From (25) we know \( P^{(I)}(0|0) = \tilde{P}^{(II)}(0|0), \) then from (25), (26) and (32), we know (29) and (30) hold. And from (23), (24) and (33), we know (27) and (28) hold.

The optimal fixed point recursive Kalman smoother of system formed by (6) and (15) is as following
\[ \hat{x}^{(I)}(t | t + N) = \hat{x}^{(I)}(t | t + N - 1) + K^{(I)}(t | t + N)\times e^{(I)}(t + j), \quad N > 0 \]  
(34)

This equation is iterated \( N \) times and the optimal fixed-lag non-recursive Kalman smoother is yielded as
\[ \hat{x}^{(I)}(t | t + N) = \hat{x}^{(I)}(t | t) + \sum_{j=0}^{N} K^{(I)}(t | t + j)e^{(I)}(t + j), \]  
(35)
\( i = I, II, \)

variance matrix \( P^{(I)}(t | t + N) = \tilde{E}[\hat{x}^{(I)}(t | t + N)\hat{x}^{(I)T}(t | t + N)] \) of estimation error \( \hat{x}^{(I)}(t | t + N) = x(t) - \hat{x}^{(I)}(t | t + N) \) is
\[ P^{(I)}(t | t + N) = P^{(I)}(t | t) - \sum_{j=0}^{N} K^{(I)}(t | t + j)\times Q^{(I)}(t | t + j), \]  
(36)
where smoothing gain matrix is
\[ K^{(I)}(t | t + N) = P^{(I)}(t | t - 1)\prod_{j=0}^{N-1} [W^{(I)}(t | t + j)]H^{(I)T} \]  
(37)
\[ \Psi_{p}^{(i)}(t) = \Phi[I_{p} - K_{j}^{(i)}(t)H_{j}^{(i)}] \]  

\[ Q_{j}^{(i)}(t + N) = [H_{j}^{(i)}P_{j}^{(i)}(t + N)|t + N - 1]H_{j}^{(i)T} + R_{j}^{(i)} \]  

Smoothing gain matrix introduced from (22), (37) and (38) can also be computed or written as

\[ K_{j}^{(i)}(t + N) = P_{j}^{(i)}(t)|t\Phi^{-1} \prod_{j=1}^{N} \Psi_{p}^{(i)}(t + j)|H_{j}^{(i)T} \times Q_{j}^{(i-1)}(t + N) \]  

**Theorem 2.** The Kalman smoothers and error variance matrix of two fusion methods yielded by (34)-(40) are identical in numeric values, i.e.,

\[ \hat{x}^{(i)}(t + N) = \hat{x}^{(i)(t + N)} \quad N > 0, \forall t \]  

\[ P^{(i)}(t + N) = P^{(ii)}(t + N) \quad N > 0, \forall t \]

Proof from (20) and (32), we have

\[ K_{j}^{(i)}(t)H_{j}^{(i)} = K_{j}^{(ii)}H_{j}^{(ii)} \]

and then from (38), we have

\[ \Psi_{p}^{(i)}(t) = \Psi_{p}^{(ii)}(t) \]

From (19), we introduce

\[ H_{j}^{(i)T}Q_{j}^{(i-1)}(t)e^{(i)}(t) = P^{(i-1)}(t - 1)K_{j}^{(i)}(t)e^{(i)}(t) \]

applying (18) and (20), we have

\[ K_{j}^{(i)}(t)e^{(i)}(t) = P^{(i)}(t + 1|t)H_{j}^{(i)T}\hat{x}^{(i)}(t - 1) - P^{(i)}(t|t)H_{j}^{(i)T}R_{j}^{(i-1)}H_{j}^{(i)}\hat{x}^{(i)}(t - 1) \]

then applying Theorem 1, (32) and (33), (37), (44) we have

\[ K_{j}^{(i)}(t)e^{(i)}(t) = K_{j}^{(ii)}(t)e^{(ii)}(t) \]

\[ H_{j}^{(i)T}Q_{j}^{(i-1)}(t)e^{(i)}(t) = H_{j}^{(ii)T}Q_{j}^{(i-1)}(t)e^{(ii)}(t) \]

\[ K^{(i)}(t + j)e^{(i)}(t + j) = K^{(ii)}(t + j|t + j)e^{(ii)}(t + j) \]

\[ K^{(i)}(t + j)Q_{j}^{(i)}(t + j)K^{(i)T}(t + j + 1) = K^{(ii)}(t + j)Q_{j}^{(ii)}(t + j)K^{(ii)T}(t + j + 1) \]

Thus using (35), (36) and Theorem 1 yields that (41) and (42) hold.

Combine \( N - 1 \) times iteration of (1) and (15) into a new system, and the new system’s \( N \)-step Kalman predicator and predicating error variance matrix are as following

\[ \hat{x}^{(i)}(t + N|t) = \Phi^{x_{i}}\hat{x}^{(i)}(t + 1|t) + \sum_{j=1}^{N} \Phi^{x_{i}} Bu(t + j - 1) \]

\[ \Phi^{0}(t + N|t) = \Phi^{x_{i}}P(t + 1|t)\Phi^{x_{i}T} + \sum_{j=2}^{N} \Phi^{x_{i}} R_{j}^{x_{i}}\Phi^{x_{i}T} \]

**Theorem 3.** The \( N \)-step Kalman predicators and error variance matrix of two fusion methods yielded by (50) and (51) are equivalent in numeric values, i.e.,

\[ \hat{x}^{(i)}(t + N|t) = \hat{x}^{(ii)}(t + N|t) \quad N > 0, \forall t \]

\[ P^{(i)}(t + N|t) = P^{(ii)}(t + N|t) \quad N > 0, \forall t \]

Proof Applying Theorem 1, (50) and (51) can yield that (52) and (53) hold.

4. Selection for Full-rank Decomposition of Matrix

From the equation of (9), we can know that when using the algorithm of WMF, the full-rank decomposition of the extended measurement matrix \( H^{(i)} \) should be calculated firstly. However, the full-rank decomposition of matrix is not unique. In fact, for the equation of (9), if there is a compatible and inverted matrix of \( C \), then \( H^{(i)} = FC^{-1}H^{(i)\text{inv}}(FC)\times(C^{-1}H^{(i)}) \). When it is substituted into theorem proof, the theorem is still established. The full-rank decomposition of matrix can be calculated by both the method of elementary row transformation and Hermite standard model. When the method of elementary row transformation is used, the inverse of matrix should be calculated, while the inverse of matrix should not be calculated and only the row standard simplest form of \( H^{(i)} \) should be calculated through elementary row transformation when the method of Hermite standard model is used. When \( H^{(i)} \) is column full rank or row full rank, the following two ordinary decompositions of \( H^{(i)} \) can be directly obtained.

**Corollary 1.** In the systems (1) and (2), when all the sensors have the same measurement matrix of \( H \in R^{m \times n} \), which is row full rank, an ordinary full-rank decomposition of \( H^{(i)} \) can be obtained, which is \( H^{(i)} = e\Phi H^{(i)\text{inv}}e^{T} \), where \( e = [I_{m \times 1} \ldots I_{m \times 1}]^{T} \), and \( I_{m} \) is a unit matrix with \( m \) dimensions, then the relevant weighted measurement fusion is as the follows

\[ y^{(ii)}(t) = (e^{T}R^{(i)\text{inv}}e)^{-T}e^{T}R^{(i)\text{inv}}y^{(i)}(t) \]

\[ v^{(i)}(t) = (e^{T}R^{(i)\text{inv}}e)^{-T}e^{T}R^{(i)\text{inv}}v^{(ii)}(t) \]

The new fusion measurement equation is

\[ y^{(ii)}(t) = Hx(t) + v^{(ii)}(t) \]

\[ R^{(ii)} = E(v^{(ii)}(t)v^{(ii)T}(t)) = (e^{T}R^{(i)}e)^{-1} \]
It is more specific that when $R(I) = \text{block-diag} [R_1, \ldots, R_L]$, i.e., all sensors measurement noises are inter-independent from one another, then the above equation can be conformed into

$$y(I)(t) = \left[\sum_{j=1}^{L} R_j^{-1}\right]^{-1} \left[\sum_{j=1}^{L} R_j^{-1} y_j(t) \right]$$

(58)

$$v(I)(t) = \left[\sum_{j=1}^{L} R_j^{-1}\right]^{-1} \left[\sum_{j=1}^{L} R_j^{-1} v_j(t) \right]$$

(59)

$$R(I) = \left[\sum_{j=1}^{L} R_j^{-1}\right]^{-1}$$

(60)

**Corollary 2.** When the extended measurement matrix of $H(I)$ in systems of (1) and (2) is column full rank, an ordinary full-rank decomposition of $H(I)$ can be obtained, which is $H(I) = H(I)_{11}$. Then the relevant weighted measurement fusion is

$$y(I)(t) = (H(I)^T R(I)^{-1} H(I))^{-1} H(I)^T R(I)^{-1} y(I)(t)$$

(61)

$$v(I)(t) = (H(I)^T R(I)^{-1} H(I))^{-1} H(I)^T R(I)^{-1} v(I)(t)$$

(62)

The new fusion measurement equation is

$$y(I)(t) = x(t) + v(I)(t)$$

(63)

$$R(I) = E(v(I)(t)v(I)^T(t)) = (H(I)^T R(I)^{-1} H(I))^{-1}$$

(64)

Corollaries of 1 and 2 are just the results in [3-7], which show that the weighted measurement fusion estimation algorithm presented in this paper is general, so it has broader application areas and more important application values.

**5. Simulation Analysis**

Consider 3-sensor target tracking system with color measurement noise

$$x(t+1) = \Phi x(t) + \Gamma w(t)$$

(65)

$$z_i(t) = H_{oi} x(t) + \eta_i(t) + b_i, \quad i = 1, 2, 3$$

(66)

$$\eta_i(t+1) = a_i \eta_i(t) + \xi_i(t),$$

(67)

$$\Phi = \begin{bmatrix} 1 & T_0 \\ 0 & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.5T_0^2 \\ T_0 \end{bmatrix},$$

$$H_{oi} = H_{o2} = [1 \ 0], \ H_{o3} = [0 \ 1] ,$$

where $T_0 = 1$ is the sampled period, $x(t) = [x_1(t), x_2(t)]^T$, $z_i(t) = [z_1(t), z_2(t)]^T$, $a_i = i/10$, $b_i = i$, $i = 1, 2, 3$, $w(t)$ and $\xi_i(t)$ are the independent Gaussian white noise with zero mean and variance $Q_w = 1$, $\sigma_{\xi}^2 = 4i$ ($i = 1, 2, 3$). Introducing backward shift operator $q^{-1}$, then (67) can be written as

$$\eta_i(t+1) = \xi_i(t),$$

Setting $t = t+1$ in (66), make (66) pre-multiplied by $(1-a_i q^{-1})$, and using (66) and (67) yield a new measurement equation with white measurement noise

$$y_i(t) = H_i x(t) + v_i(t)$$

where

$$y_i(t) = \bar{z}_i(t+1) - a_i \bar{z}_i(t)$$

$$v_i(t) = \xi_i(t)$$

$$H_i = H_{oi} - a_i H_{oi}$$

Simulation results are shown in Fig. 1 - Fig. 4, from which we can see that centralized fusion and weighted measurement fusion have the same estimation accuracy.

**6. Conclusions**

This paper has presented a new weighted measurement fusion algorithm by using full-rank decomposition of matrix and weighted least squares theory. It proves that it has global optimality and its computation burden is reduced greatly. Meanwhile, the method of ordinary decomposition is used to prove that the methods presented in [3-7] are just special cases of the algorithm presented in this paper.
Fig. 3. Filtering error of position using two algorithms.

Fig. 4. Filtering error of speed using two algorithms.

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