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Long Term Numerical Optimal Controls on a Class of Integro-differential Equations with Weakly Singular Kernels

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Abstract: This article presents a method of optimizing dynamic systems governed by integro-differential equations of the second kind with weakly singular kernels; in such systems, the time range is infinite and the controls may be delayed. Performance assessments involve measuring the square distance between tracking functions and states and measuring the energy of controls, and measuring the deviation of the final states from the tracking functions. We report the results of typical examples.

Keywords: Integro-differential equations, Weakly singular, Tracking functions, Optimal controls.

1. Introduction

Optimal control problems have been studied in numerous fields. Scholars have described numerical optimization problems involving constraints and weakly singular integro-differential equations of the first [1-4] and second [5, 6] kinds. This paper focuses on minimum energies in tracking functions and on controls in dynamic systems governed by integro-differential equations of the second kind, in which the controls for the dynamic systems are assumed to introduce a delay. This paper is organized as follows: Section 2 presents a dynamic system modeled using weakly singular integro-differential equations of the second kind. Section 3 describes an algorithm for optimal delay control. Section 4 provides numerical results for selected examples, and finally, Section 5 summarizes this paper.

2. Weakly Singular Equations

Consider an integro-differential equation of the second kind:

$$\frac{d}{dt}x(t)+\frac{d}{dt}x(t-b)+\frac{d}{dt}Dx_t=u(t,\sigma), \quad (1)$$

where t, b and σ are ≥ 0, and the initial conditions are

$$x(s)=\varphi(s), u(s)=0, s\leq 0 \quad (2)$$

In Eq. (1), b and σ are constants, with $\sigma \ll b$.
The Operator D satisfies

$$Dx_t=\int_{-b}^{0} g(s)x(t+s)ds$$

The second part of the integrand represents

$$x_t(s)=x(t+s),$$

whereas the first part is a weakly singular function,

$$g(s)\in L_1[-b,\ 0],$$

which is integrable, positive, nondecreasing, and weakly singular at $s=0$. In particular, the simpler form of $g(s)=|s|^{-p}$ (with a similar property to

$$g(s) = \left(1 - \frac{1}{s}\right)^{0.5},$$

in the aeroelasticity problem [2]) was considered, where $0 < p < 1$.

If the integral Dx_0 exists, integrating Equation (1) yields the following equation:

$$x(t) + \varphi(t-b) + Dx_t = Dx_0 + \varphi(0) + \\ +\varphi(-b) + \int_0^t u(\tau,\sigma)\, d\tau, \text{ for } 0 < t \le b$$

This equation can be represented as a Volterra integral equation of the second kind provided that the function

$$Dx_t = \int_{-b}^{0} |s|^{-p} x(t+s) ds$$

is absolutely continuous with respect to $t > 0$ and that the product of the kernel and initial function, expressed as $g(\cdot)\varphi(\cdot)$ belongs to $L_1[-b, 0]$.

Without loss of generality, we assume $b = 1$ and use the equation

$$\frac{d}{dt}x(t) + \frac{d}{dt}x(t-1) + \\ +\frac{d}{dt}\int_{-1}^{0} |s|^{-p} x(t+s) ds = u(t,\sigma), \\ 0 \le t \le 1, 0 \le \sigma \ll 1,$$

with the initial data $x(s) = \varphi(s)$, $u(s) = 0$, $s \le 0$, where the control function $u(t,\sigma)$ is a locally integrable function.

The proposed method can be applied to more general cost functions. This paper compares two typical cost functions:

$$\Phi_1 = \alpha_1 \int_0^{b_1} (x(t) - \eta(t))^2 dt + \\ +\alpha_2 \int_0^{b_1} u(t)^2 dt,$$

and

$$\Phi_2 = \alpha_1 (x(b_1) - \eta(b_1))^2 + \alpha_2 \int_0^{b_1} (x(t) - \\ -\eta(t))^2 dt + \alpha_3 \int_0^{b_1} u(t)^2 dt,$$

where $\eta(t)$ is a target function and α_1, α_2, and α_3 are nonnegative constants, whose sum is 1 in each corresponding cost function, with $b_1 \to \infty$.

3. Numerical Method

We discretize Equation (1) with the delay case $u(t,\sigma) = u(t-\sigma)$ and the cost functions Φ_i, where $i = 1, 2$ for $b_1 = b = 1$, to construct a linear system with unknowns as states and controls. The space mesh points (corresponding to the s variable) are discretized as $-1 = \tau_n < \tau_{n-1} < \cdots < \tau_1 < \tau_0 = 0$, and a new variable ξ is defined as follows:

$$\xi(t,s) = x(t+s), -1 \le s \le 0, 0 < t \tag{3}$$

Equation (1) can then be reformulated as the first-order hyperbolic equation,

$$\frac{\partial}{\partial t}\xi(t,s) = \frac{\partial}{\partial s}\xi(t,s), -1 \le s \le 0, \tag{4}$$

with the condition

$$\frac{\partial}{\partial t}\xi(t,0) + \frac{\partial}{\partial t}\xi(t,-1) + \\ + \int_{-1}^{0} |s|^{-p} \frac{\partial}{\partial s}\xi(t,s) ds = u(t-\sigma) \tag{5}$$

Next, we assume that the solution to Equations (4) and (5) has the form

$$\xi(t,s) = \sum_{i=0}^{n} \kappa_i(t) B_i(s), \tag{6}$$

where the basis $B_i(s)$, $i = 0, \cdots, n$, is expressed as follows:

$$B_i(s) = \begin{cases} \frac{1}{(\tau_i - \tau_{i+1})}(s - \tau_{i+1}) & s \in [\tau_{i+1}, \tau_i], \\ \frac{1}{(\tau_{i-1} - \tau_i)}(\tau_{i-1} - s) & s \in [\tau_i, \tau_{i-1}], \\ 0 & \text{otherwise} \end{cases}$$

Here, $B_i(s)$, $i = 0, 1, \cdots, n$, represents piecewise linear functions. When the special form of ξ in Equation (6) is substituted into Equations (4) and (5), the governing equations for $\kappa_i(t), i = 0, \cdots, n$, become the following:

$$\frac{d}{dt}\kappa_i(t) = \frac{1}{\delta_i}\left(\kappa_{i-1}(t) - \kappa_i(t)\right), \\ i = 1, \cdots, n, \tag{7}$$

$$\frac{d}{dt}\sum_{i=0}^{n} \kappa_i(t) B_i(0) + \frac{d}{dt}\sum_{i=0}^{n} \kappa_i(t) B_i(-1) + \\ + \int_{-1}^{0} |s|^{-p} \sum_{i=0}^{n} \kappa_i(t) \frac{d}{ds} B_i(s) ds u(t-\sigma), \tag{8}$$

where $\delta_i = \tau_{i-1} - \tau_i > 0$, for $i = 1, \cdots, n$.

For time t, the discretization contains $T^0, T^1, \cdots, T^m$, for $0 = T^0 < T^1 < \cdots < T^m = 1$.

We define

$$\Delta^k = T^{k+1} - T^k, \text{ for } k = 0, \cdots, m-1$$

By assuming $\alpha_i^k = \kappa_i(T^k)$, for $i = 0, 1, \cdots, n$, and $k = 0, \cdots, m$.

We can express Equations (7) and (8) as follows:

$$\frac{1}{\Delta^k}(\alpha_i^{k+1} - \alpha_i^k) = \frac{1}{\delta_i}(\alpha_{i-1}^k - \alpha_i^k), \tag{9}$$

$$\frac{1}{\delta_1}\alpha_0^{k+1} - \frac{1}{\delta_1}\alpha_1^{k+1} + \frac{1}{\delta_{n-1}}\alpha_{n-1}^{k+1} - \tag{10}$$

$$-\frac{1}{\delta_{n-1}}\alpha_n^{k+1}+\sum_{i=1}^{n}\frac{g_i}{\delta_i}\left(\alpha_{i-1}^{k+1}-\alpha_i^{k+1}\right)= \\ =u(T^{k+1}-\sigma),$$

where $i=1,\dots,n$, $k=0,\dots,m-1$, and

$$g_i=\int_{\tau_i}^{\tau_{i-1}}|s|^{-p}ds$$

Furthermore, we assume a uniform mesh for both space and time. The mesh points are τ_i, $i=0,\cdots,n$, and T^k, $k=0,\cdots,m$. Specifically, we have $\tau_i=-\frac{i}{n}$, and $T^k=\frac{k}{m}$ for some positive integers n and m. Thus, $\Delta^k=\frac{1}{m}$ and $\delta_i=\frac{1}{n}$ for $k=0,\cdots,m-1$, and $i=1,\dots,n$. Setting $m=n$ yields the relation $\Delta^k=\delta_i=\frac{1}{n}$, for $k=0,\cdots,n-1$, and $i=1,\cdots,n$. Equations (9) and (10) then yield the following system:

$$\alpha_i^{k+1}=\alpha_{i-1}^{k}, \tag{11}$$

$$\frac{1}{\delta_1}(\alpha_0^{k+1}-\alpha_1^{k+1})+\frac{1}{\delta_{n-1}}(\alpha_{n-1}^{k+1}- \\ -\alpha_n^{k+1})+\sum_{i=1}^{n}\frac{1}{\delta_i}\left(\alpha_{i-1}^{k+1}-\alpha_i^{k+1}\right)\cdot \\ \frac{1}{1-p}[-\tau_{i-1})1-p+(-\tau_i)^{1-p}= \\ =u(T^{k+1}-\sigma), \tag{12}$$

for $i=1,\cdots,n$, and $k=0,\cdots,n-1$.

After defining the corresponding constants $c_0,c_1,\cdots,c_n$, we can simplify Equation (12) as follows:

$$\alpha_0^{k+1}c_0+\alpha_0^{k}c_1+\cdots+\alpha_0^{0}c_{k+1}+\cdots+ \\ +\alpha_{n-k-1}^{0}c_n=u_{k+1-j(\sigma)}, \\ k=0,\cdots,n-1,\text{ and }0\le j(\sigma)\le n \tag{13}$$

The relation between the solution $x(t)$ and α_0^j for $j=1,\cdots,n$ is described as follows: Because $\xi(t,s)=x(t+s)$ for $-1\le s\le 0$, $0<t$ and $\xi(t,s)=\sum_{i=0}^{n}\kappa_i(t)B_i(s)$, we can obtain $x(\bar{t})$ for $0<\bar{t}\le 1$ by using the following formula:

$$x(T^j)=\xi(T^j,0)= \\ =\sum_{l=0}^{n}\kappa_l(T^j)\,B_l(0)=\kappa_0(T^j)=\alpha_0^j, \\ \text{for } j=1,\cdots,n$$

The discretized forms of the cost functions

$$\Phi_1=\alpha_1\int_0^1(x(t)-\eta(t))^2dt+ \\ +\alpha_2\int_0^1 u(t)^2dt,$$

and

$$\Phi_2=\alpha_1(x(1)-\eta(1))^2+\alpha_2\int_0^1(x(t)- \\ -\eta(t))^2dt+\alpha_3\int_0^1 u(t)^2dt$$

are as follows:

$$\Phi_{1D}=\alpha_1\frac{1}{n}\sum_{i=1}^{n}\left(\alpha_0^i-\eta(T^i)\right)^2+ \\ +\alpha_2\frac{1}{n}\sum_{i=1}^{n}{u_i}^2,$$

and

$$\Phi_{2D}=\alpha_1(\alpha_0^n-\eta(T^n))^2+ \\ +\alpha_2\frac{1}{n}\sum_{i=1}^{n}\left(\alpha_0^i-\eta(T^i)\right)^2+\alpha_3\frac{1}{n}\sum_{i=1}^{n}{u_i}^2$$

We can take the first derivatives of Φ_{iD}, $i=1,2$, with respect to u_i, $i=1,\cdots,n$, and set them to zero, to yield the following corresponding equations (14) and (15):

$$\begin{gathered}\alpha_1\begin{bmatrix}aa(1)\cdot\alpha_0^{1+j(\sigma)}+aa(2)\cdot\alpha_0^{2+j(\sigma)}+ \\ +\cdots+aa\left(n-j(\sigma)\right)\cdot\alpha_0^n\end{bmatrix}+\alpha_2\cdot u_1 \\ = \\ =\alpha_1\begin{bmatrix}\eta\left(T^{1+j(\sigma)}\right)\cdot aa(1)+\cdots+\eta(T^n)\cdot \\ \cdot aa(n-j(\sigma))\end{bmatrix}, \\ \vdots \\ \alpha_1\begin{bmatrix}aa(1)\cdot\alpha_0^{j+j(\sigma)}+aa(2)\cdot\alpha_0^{j+j(\sigma)+1}+ \\ +\cdots+aa(n-j-j(\sigma)+1)\cdot\alpha_0^n\end{bmatrix}++\alpha_2\cdot \\ u_j=\alpha_1\left[\eta\left(T^{j+j(\sigma)}\right)\cdot aa(1)+\cdots++\eta(T^n)\cdot\right. \\ \left.aa(n-j-j(\sigma)+1)\right], \\ \vdots \\ \alpha_1\cdot\alpha_0^n\cdot aa(1)+\alpha_2\cdot u_{n-j(\sigma)}=\alpha_1\cdot \\ \cdot\eta(T^n)\cdot aa(1), \\ \alpha_2\cdot u_{n-j(\sigma)+1}=0, \\ \alpha_2\cdot u_{n-j(\sigma)+2}=0, \\ \vdots \\ \alpha_2\cdot u_n=0,,\end{gathered} \tag{14}$$

and

$$\begin{gathered}\alpha_2\begin{bmatrix}aa(1)\cdot\alpha_0^{1+j(\sigma)}+aa(2)\cdot\alpha_0^{2+j(\sigma)}+\cdots+ \\ +aa\left(n-j(\sigma)\right)\cdot\alpha_0^n\end{bmatrix}+ \\ +\alpha_3\cdot u_1=n\alpha_1\eta(T^n)aa\left(n-j(\sigma)\right)+ \\ +\alpha_2\begin{bmatrix}\eta\left(T^{1+j(\sigma)}\right)\cdot aa(1)+\cdots+ \\ +\eta(T^n)\cdot aa(n-j(\sigma))\end{bmatrix}, \\ \vdots \\ = \\ \alpha_2\begin{bmatrix}aa(1)\cdot\alpha_0^{j+j(\sigma)}+aa(2)\cdot\alpha_0^{j+j(\sigma)+1}+\cdots+ \\ +aa(n-j-j(\sigma)+1)\cdot\alpha_0^n\end{bmatrix}+ \\ +\alpha_3\cdot u_j=n\alpha_1\eta(T^n)aa(n-j-j(\sigma)+1)+ \\ \alpha_2\left[\eta(T^{j+j(\sigma)})\cdot aa(1)+\cdots+\eta(T^n)\cdot aa(n-j-\right. \\ \left.j(\sigma)+1)\right], \\ \vdots \\ \alpha_2\cdot\alpha_0^n\cdot aa(1)+\alpha_3\cdot u_{n-j(\sigma)}=(n\alpha_1+\alpha_2)\cdot \\ \eta(T^n)\cdot aa(1), \\ \alpha_3\cdot u_{n-j(\sigma)+1}=0, \\ \alpha_3\cdot u_{n-j(\sigma)+2}=0, \\ \vdots \\ \alpha_3\cdot u_n=0,\end{gathered} \tag{15}$$

where $aa(1) = \frac{1}{c_0}, aa(2) = -\frac{c_1}{c_0} \cdot aa(1), \ldots aa(j) = -\frac{c_1}{c_0} aa(j-1) - \frac{c_2}{c_0} \cdot aa(j-2) - \cdots - \frac{c_{j-1}}{c_0} \cdot aa(1)$, $\ldots \quad aa(n-j(\sigma)) = -\frac{c_1}{c_0} aa(n-j(\sigma)-1) - \frac{c_2}{c_0} \cdot aa(n-j(\sigma)-2) - \cdots - \frac{c_{n-j(\sigma)-1}}{c_0} \cdot aa(j(\sigma)+1)$.

Systems (13) and (14) or (13) and (15) can be represented as $[A][x] = [b]$, where the vector $[x]$ consists of the unknowns $\alpha_0^j, j = 1, \cdots, n$, and u_k, $k = 1, \cdots, n$. The structure of the matrices $[A]$ is expressed as follows:

$$\begin{bmatrix} c_0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ c_1 & c_0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & \cdots & c_0 & 0 & \cdots & -1 & 0 \\ 0 & \alpha_1 aa(1) & \cdots & \alpha_1 aa(n-j(\sigma)) & \alpha_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \vdots & 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \alpha_1 aa(1) & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & 0 & 0 & & & \alpha_2 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \alpha_2 \end{bmatrix}_{2n\times 2n},$$

and

$$\begin{bmatrix} c_0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ c_1 & c_0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & \cdots & c_0 & 0 & \cdots & -1 & 0 \\ 0 & \alpha_2 aa(1) & \cdots & \alpha_2 aa(n-j(\sigma)) & \alpha_3 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \vdots & 0 & \alpha_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \alpha_2 aa(1) & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & 0 & 0 & & & \alpha_3 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \alpha_3 \end{bmatrix}_{2n\times 2n}$$

Moreover, the vectors $[b]$ are expressed as follows:

$$\begin{bmatrix} -\alpha_0^0 c_1 - \alpha_1^0 c_2 - \cdots - \alpha_{n-1}^0 c_n + u_{1-j(\sigma)} \\ \vdots \\ -\alpha_0^0 c_{j(\sigma)+1} - \alpha_1^0 c_{j(\sigma)+2} - \cdots - \alpha_{n-j(\sigma)-1}^0 c_n + u_{2-j(\sigma)} \\ \vdots \\ -\alpha_0^0 c_n \\ \alpha_1\left[\eta(T^{1+j(\sigma)})aa(1) + \cdots + \eta(T^n)aa(n-j(\sigma))\right] \\ \alpha_1\left[\eta(T^{2+j(\sigma)})aa(1) + \cdots + \eta(T^n)aa(n-j(\sigma)-1)\right] \\ \vdots \\ \alpha_1\left[\eta(T^{j+j(\sigma)})aa(1) + \cdots + \eta(T^n)aa(n-j(\sigma)-j+1)\right] \\ \vdots \\ \alpha_1\eta(T^n)aa(1) \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{2n\times 1}$$

and

$$\begin{bmatrix} -\alpha_0^0 c_1 - \alpha_1^0 c_2 - \cdots - \alpha_{n-1}^0 c_n + u(T^1 - \sigma) \\ \vdots \\ -\alpha_0^0 c_{j(\sigma)+1} - \alpha_1^0 c_{j(\sigma)+2} - \cdots - \alpha_{n-j(\sigma)-1}^0 c_n + u(T^2 - \sigma) \\ \vdots \\ -\alpha_0^0 c_n \\ n\alpha_1\eta(T^n)aa(n-j(\sigma)) + \alpha_2\left[\eta(T^{j(\sigma)+1})aa(1) + \cdots + \eta(T^n)aa(n-j(\sigma))\right] \\ n\alpha_1\eta(T^n)aa(n-j(\sigma)-1) + \alpha_2\left[\eta(T^{j(\sigma)+2})aa(1) + \cdots + \eta(T^n)aa(n-j(\sigma)-1)\right] \\ \vdots \\ n\alpha_1\eta(T^n)aa(n-j(\sigma)-j+1) + \alpha_2\left[\eta(T^{j(\sigma)+j})aa(1) + \cdots + \eta(T^n)aa(n-j(\sigma)-j+1)\right] \\ \vdots \\ n\alpha_1\eta(T^n)aa(1) + \alpha_2\eta(T^n)aa(1) \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{2n\times 1}$$

The positions of value -1 in the upper right block of matrices $[A]$ depend on the delay σ; for $\sigma = 0$, the block matrix is diagonal. Controls in the upper part of $[b]$ depend on σ from specific previous controls.

For $b_1 \gg 1$, $\Phi_1 = \alpha_1 \int_0^{b_1} (x(t) - \eta(t))^2 dt + \alpha_2 \int_0^{b_1} u(t)^2 dt$, and

$$\Phi_2 = \alpha_1 \big(x(b_1) - \eta(b_1)\big)^2 + \alpha_2 \int_0^{b_1} (x(t) - -\eta(t))^2 dt + \alpha_3 \int_0^{b_1} u(t)^2 dt$$

The cost calculations are performed using a step-based method involving the intervals $[0,1], [1,2], \cdots \to \infty$. In each step, the state and control are updated using the results of the previous step. The final cost is calculated as the sum of the costs derived in each step.

4. Numerical Examples

We consider examples with $p = 0.5$; different initial conditions, namely $\varphi(s), -1 \le s \le 0$; and different target functions, namely $\eta(t)$, $0 \le t \le b_1$. The combination of α constants in the cost functions is adjusted for different parameters. The cost of real-time controls ($\sigma = 0$) is provided for comparison in each example to ensure accuracy.

Examples 1-4 are the extensions from a previous paper [7]. Examples 5-8 are the results for the assessment Φ_2.

Example 1: $n = 100, \varphi(s) = s^2, -1 \le s \le 0$; $u(s) = 0, -1 \le s \le 0; \eta(t) = \cos(t), 0 \le t \le 10$; $(\alpha_1, \alpha_2) = (0,1), (0.1, 0.9), (0.3, 0.7)$, and $(0.5, 0.5)$.

The total costs are shown in Table 1.

Table1. Total costs for Example 1.

(α_1, α_2)	$\sigma = 0$	$\sigma = 0.2$	$\sigma = 0.5$
$(0, 1)$	0	0	0
$(0.1, 0.9)$	0.0194	0.0194	0.0194
$(0.3, 0.7)$	0.0581	0.0582	0.0582
$(0.5, 0.5)$	0.0975	0.0980	0.0976

Example 2: $n = 100, \varphi(s) = s, -1 \le s \le 0; u(s) = 0, -1 \le s \le 0; \eta(t) = t, 0 \le t \le 10$; $(\alpha_1, \alpha_2) = (0,1), (0.1, 0.9), (0.3, 0.7)$, and $(0.5, 0.5)$.

The total costs are shown in Table 2.

Table 2. Total costs for Example 2.

(α_1, α_2)	$\sigma = 0$	$\sigma = 0.2$	$\sigma = 0.5$
$(0, 1)$	0	0	0
$(0.1, 0.9)$	0.1938	0.1961	0.1978
$(0.3, 0.7)$	0.5515	0.5756	0.5931
$(0.5, 0.5)$	0.8519	0.9270	0.9879

Example 3: $n = 100, \varphi(s) = \sin(s), -1 \le s \le 0$; $u(s) = 0, -1 \le s \le 0; \eta(t) = t, 0 \le t \le 10$; $(\alpha_1, \alpha_2) = (0,1), (0.1, 0.9), (0.3, 0.7)$, and $(0.5, 0.5)$.

The total costs are shown in Table 3.

Table 3. Total costs for Example 3.

(α_1, α_2)	$\sigma = 0$	$\sigma = 0.2$	$\sigma = 0.5$
$(0, 1)$	0	0	0
$(0.1, 0.9)$	0.1938	0.1961	0.1978
$(0.3, 0.7)$	0.5515	0.5756	0.5931
$(0.5, 0.5)$	0.8519	0.9270	0.9879

Example 4: $n = 100, \varphi(s) = 1 - s, -1 \le s \le 0$; $u(s) = 0, -1 \le s \le 0; \eta(t) = \sin(t), 0 \le t \le 10$; $(\alpha_1, \alpha_2) = (0,1), (0.1, 0.9), (0.3, 0.7)$, and $(0.5, 0.5)$.

The total costs are shown in Table 4.

Table 4. Total costs for Example 4.

(α_1, α_2)	$\sigma = 0$	$\sigma = 0.2$	$\sigma = 0.5$
$(0, 1)$	0	0	0
$(0.1, 0.9)$	0.1984	0.2013	0.2036
$(0.3, 0.7)$	0.5516	0.5807	0.6049
$(0.5, 0.5)$	0.8223	0.9101	0.9919

Example 5: $n = 100, \varphi(s) = 0, u(s) = 0, s \le 0$; $\eta(t) = \cos(t), 0 \le t \le 20$; $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 1)$, $(0.1, 0.1, 0.8)$, $(0.1, 0.2, 0.7)$ and $(0.1, 0.3, 0.6)$.

The total costs are shown in Table 5.

Table 5. Total costs for Example 5.

$(\alpha_1, \alpha_2, \alpha_3)$	$\sigma = 0$	$\sigma = 0.2$	$\sigma = 0.5$
$(0, 0, 1)$	0	0	0
$(0.1, 0.1, 0.8)$,	0.1795	0.1821	0.1899
$(0.1, 0.2, 0.7)$	0.2175	0.2188	0.2252
$(0.1, 0.3, 0.6)$	0.2582	0.2586	0.2624

Some graphs are presented in the Appendix.

Example 6: $n = 100, \varphi(s) = s, u(s) = 0, -1 \le s \le 0; \eta(t) = 1 - t, 0 \le t \le 20; (\alpha_1, \alpha_2, \alpha_3) = (0, 0, 1), (0.1, 0.1, 0.8), (0.1, 0.2, 0.7)$ and $(0.1, 0.3, 0.6)$.

The total costs are shown in Table 6.

Table 6. Total costs for Example 6.

$(\alpha_1, \alpha_2, \alpha_3)$	$\sigma = 0$	$\sigma = 0.2$	$\sigma = 0.5$
$(0, 0, 1)$	0	0	0
$(0.1, 0.1, 0.8)$,	1.3043	1.3613	1.4856
$(0.1, 0.2, 0.7)$	1.4153	1.4951	1.6894
$(0.1, 0.3, 0.6)$	1.4830	1.5740	1.8315

Some graphs are displayed in the Appendix.

Example 7: $n = 100, \varphi(s) = \cos(s), u(s) = 0, -1 \le s \le 0; \eta(t) = t, 0 \le t \le 20; (\alpha_1, \alpha_2, \alpha_3) = (0, 0, 1), (0.1, 0.1, 0.8), (0.2, 0.2, 0.6)$ and $(0.2, 0.3, 0.5)$.

The total costs are shown in Table 7.

Table 7. Total costs for Example 7.

$(\alpha_1, \alpha_2, \alpha_3)$	$\sigma = 0$	$\sigma = 0.2$	$\sigma = 0.5$
$(0, 0, 1)$	0	0	0
$(0.1, 0.1, 0.8)$,	0.4499	0.4541	0.4549
$(0.2, 0.2, 0.6)$	0.9084	0.9299	0.9377
$(0.2, 0.3, 0.5)$	1.2780	1.3412	1.3808

Example 8: $n = 100, \varphi(s) = \cos(s)$, $u(s) = 0$, $-1 \le s \le 0$; $\eta(t) = \sin(t)$, $0 \le t \le 20$; $(\alpha_1, \alpha_2, \alpha_3) = (0,0,1), (0.1,0.1,0.8), (0.2,0.2,0.6)$ and $(0.2,0.3,0.5)$.

The total costs are shown in Table 8.

Table 8. Total costs for Example 8.

$(\alpha_1, \alpha_2, \alpha_3)$	$\sigma = 0$	$\sigma = 0.2$	$\sigma = 0.5$
$(0,0,1)$	0	0	0
$(0.1,0.1,0.8)$,	0.3850	0.3957	0.4047
$(0.2,0.2,0.6)$	0.7239	0.7641	0.8023
$(0.2,0.3,0.5)$	1.0115	1.0982	1.1880

Some graphs are presented in the Appendix.

The minimum costs derived using delay controls are founded to be typically higher than the costs derived using real-time controls. The tracking of parts results in higher costs.

5. Summary

This paper presents a method for finding the optimal controls and corresponding optimal states for weighted cost functions that emphasize different parameters and are governed by an integro-differential equation of the second kind. The effectiveness of the proposed method is tested in selected examples. The cost of the delay controls ($\sigma > 0$) is higher than that of the deterministic optimal controls ($\sigma = 0$), and tracking is found to contribute more to the total cost than are the controls.

Acknowledgements

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References

[1]. S. Chiang, Notes on the solution of a class of singular integral equations, *Chung Hua Journal of Science and Engineering*, Vol. 3, Issue 4, July 2005, pp. 89-95.

[2]. J. A. Burns, E. M. Cliff, T. L. Herdman, A state-space model for an aeroelastic system, in *Proceedings of the 22nd IEEE Conference on Decision and Control*, 1983, pp. 1074-1077.

[3]. S. Chiang, T. L. Herdman, On the infinite field of a class of weakly singular integral equations, *Far East Journal of Dynamical Systems*, Vol. 34, 2022, pp. 11-23.

[4]. F. Kappel, K. P. Zhang, On neutral functional differential equations with nonatomic difference operator, *Journal of Mathematical Analysis and Application*, Vol. 113, 1986, pp. 311-343.

[5]. S. Chiang, T. L. Herdman, Numerical algorithms for solving optimal control problems with integro-differential equations of the second kind as constraints, in *Proceedings of the IEEE International Conference on Control & Automation (ICCA'17)*, 2017, pp. 98-102.

[6]. S. Chiang, Optimization in transition between two dynamic systems governed by a class of weakly singular integro-differential equations, *Applied Mathematics*, Vol. 10, Issue 10, 2019, pp. 826-835.

[7]. S. Chiang, Numerical optimization on a class of integro-differential equations with weakly singular kernels and possible control delays, in *Proceedings of the 4th IFSA Winter Conference on Automation, Robotics and Communications for Industry 4.0/5.0 (ARCI'24)*, 2024, pp. 141-144.

Appendix

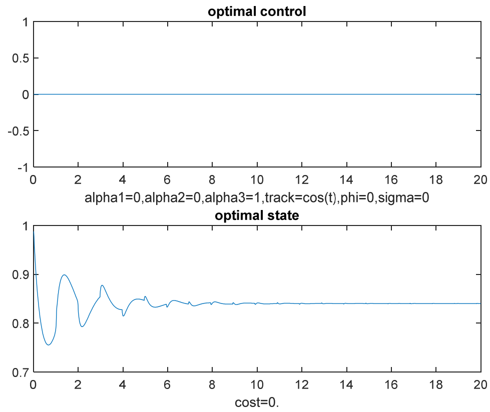

Fig. 1. Singularity p = 0.5.

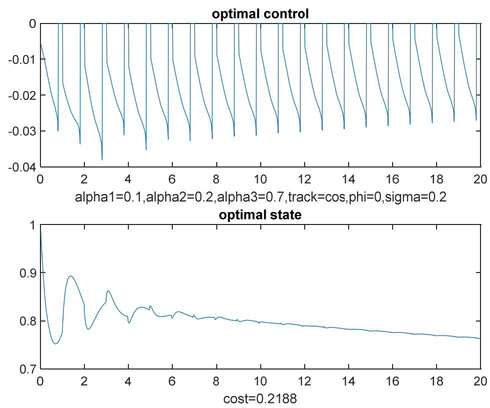

Fig. 2. Singularity p = 0.5.

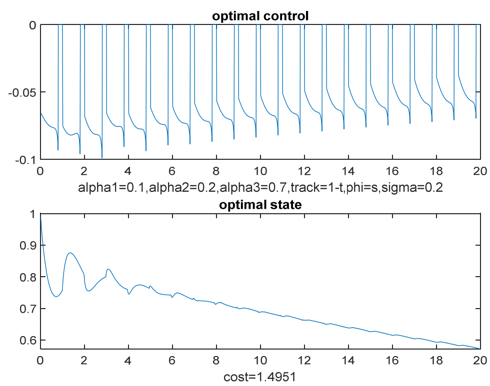

Fig. 3. Singularity p = 0.5.

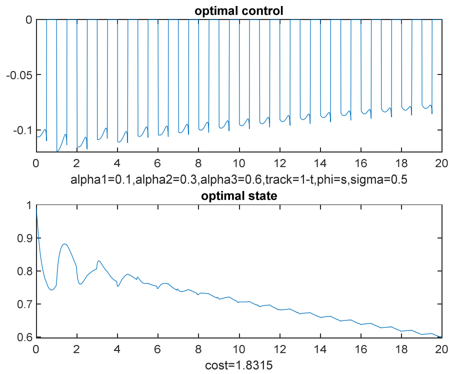

Fig. 4. Singularity p = 0.5.

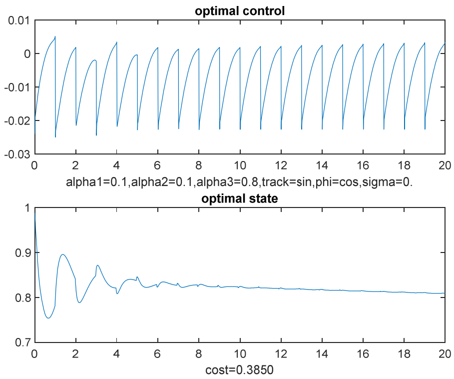

Fig. 5. Singularity p = 0.5.

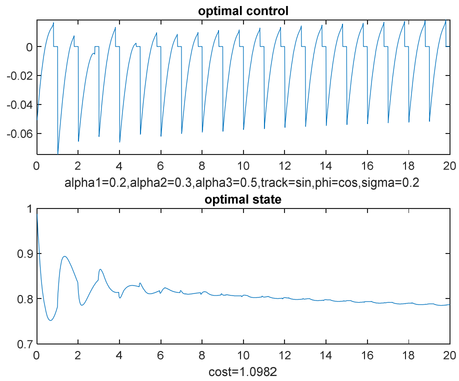

Fig. 6. Singularity p = 0.5.
