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Quantum Signal Processing and Non-Orthogonal State Detection

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Abstract: Quantum state signal for most time is Non-orthogonal signal; it's not fit to applying the classical signal processing methods to their signal analyzing and estimating. In accordance with this, we discussed a basic model for pure quantum signal processing. By combining the least-squares estimation and Positive-Operator-Valued Measure, we furthermore developed a novel quantum detection method for non-orthogonal state signal. The simulation test result in addition verified its effectiveness. *Copyright © 2013 IFSA.*

Keywords: Signal processing, Quantum detection, Non-orthogonal state, Least-square estimation, POVM.

1. Introduction

The quantum signal processing (QSP) concept is introduced by Eldar in article [1, 2], yet he barely discussed this framework of signal processing with a view on the algorithm of classic signal processing (CSP).

The classical signals are orthogonal states as a special case of quantum signal, while most quantum state signals are non-orthogonal states [9-12]. The detecting method of non-orthogonal states is shown in paper [3, 4]. Paper [5] introduced the Positive-Operator-Valued Measure (POVM) of quantum states.

To detecting quantum signal effectively, we focus on the problem of pure quantum signal processing. In this paper, Section 2 describes this model, section 3 discusses the least-squares estimation for quantum signals. Section 4 surveys the detection of non-orthogonal state signal using the least-square estimation and POVM. Section 5 presents the numerical results.

2. Quantum Signal Processing

2.1. The Model of Quantum Signal Processing

In order to build a framework of QSP, we redefine the quantum signal little differently from the classic signal. The description of classic signal (here refers to discrete time signal) is known as the time series $\{a_i\}_{-\infty}^{+\infty}$ in the Hilbert space l^{∞}, while in the case of quantum mechanics, the description of quantum signal series is

$$\{\rho_i\}_{-\infty}^{+\infty} \in \bigotimes_{k=-\infty}^{+\infty} \mathbb{C}_k^n, \tag{1}$$

where ρ_i is the density operator of n quantum bits (n-qubits) and $\bigotimes_{k=-\infty}^{+\infty} \mathbb{C}_k^n$ is the infinite dimension space, which is a tensor product of entire $\mathbb{C}_k^n$ the n-dimensional unitary spaces. Therefore, $\{\rho_i\}_{-\infty}^{+\infty}$ can

be considered as a quantum signal series in this tensor space.

The definition of the quantum signal enables us to propose the framework of QSP. As is illustrated in Fig. 1, both the input and the output signal are the classical signals. But the kernel of the QSP procedure is the quantum system. Consequently, the signals need to be transformed into quantum signals.

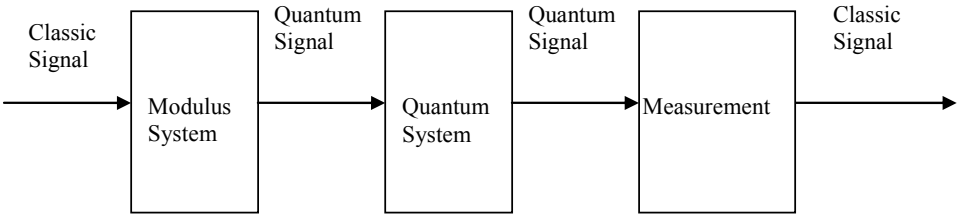

Fig. 1. The QSP model.

Meanwhile, to process classic signal of macro-world system with quantum mechanics, an approach of signal modulation is needed to transform signal from classic to quantum. And a quantum measurement must be taken on the last step of the QSP procedure so that outcome information can be read by the macro world.

In case that both input and output are classic signal, QSP can provide the advantages of parallel computing, which is exactly the purpose of quantum computation. As an example, the quantum algorithm of large number factorization proposed by Shor has won more improvement on the calculation efficiency than the conventional algorithm of classic does [5-8].

2.2. Transform for Quantum System

To stress out our study of quantum signal transform, the modulation of classic signal and the quantum measurement are not included in the following discussion. Then the transform which describes the quantum system in mathematics is showed up.

For an input series $\{\rho_i\}_{-\infty}^{+\infty}$ of a quantum system, assume the output series to be $\{\widetilde{\rho}_i\}_{-\infty}^{+\infty}$. If for each ρ_i, the quantum system doesn't cause the correlation between ρ_i and ρ_j, $i \neq j$, that is, the output $\rho_i^{'}$ and $\rho_j^{'}$ still remain independent with each other, then we can give a equation of the relationship between input and output signal according to the quantum operations formalism [5].

$$\rho_i^{'} = \varepsilon(\rho_i) = tr_{env}\left(U(\rho_i \otimes \rho_{env})U^{+}\right). \quad (2)$$

Yet a general quantum system will cause the association of the input signal elements, which makes the states of every element entangled. In this case, the quantum system should be presented by $\widetilde{U}$, which is the operator on the discrete infinite dimensional space $\bigotimes_{k=-\infty}^{+\infty} \mathbb{C}_k^n$.

Let $\{\rho_i\}_{-\infty}^{+\infty}$ be the input series, so the density matrix is

$$\widetilde{\rho} = \cdots \rho_{-k} \otimes \rho_{-k+1} \otimes \cdots \otimes \rho_0 \otimes \rho_1 \otimes \cdots \rho_k \cdots = \bigotimes_{k=-\infty}^{\infty} \rho_k \quad (3)$$

Then the density matrix of output series is

$$\widetilde{\rho}^{'} = \varepsilon(\widetilde{\rho}) = tr_{env}\left[\widetilde{U}\left(\widetilde{\rho} \otimes \rho_{env}\right)\widetilde{U}^{\dagger}\right].$$

As can be seen, the states of every element of output series have already entangled.

2.3. Frequency Domain Transform of Quantum Signal

As we have mentioned above, the quantum signal essentially is the tensor in Hilbert space. There is still a problem that the choice of different basis results in different representation of the tensor. And QSP also has the similar transform as frequency domain (FD) transform in the field of CSP.

Concerning the quantum discrete Fourier transform of pure states

$$|j> \rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi jk/N} |k>, \quad (4)$$

as $|j>$ refers to the basis state, and for the general states we have

$$\sum_{j=0}^{N-1} x_j |j> \rightarrow \sum_{k=0}^{N-1} y_k |k>,$$

where

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi jk/N}.$$

Quantum discrete Fourier transform is a key part of quantum algorithm, such as large number factorization and many other interesting quantum algorithms. And one of its important roles is phase estimation, that can approximate the eigenvalues of unitary operator on some specified occasions [7]. But here, transforming quantum signal from time domain (TD) to the FD is its another important application. In order to get the formalism of this transform, we need to extend the concept of pure states to mixed states. For the quantum series, the tensor in Hilbert space is expressed as $\overset{N-1}{\underset{k=0}{\otimes}}\rho_k$, and its basis is $\{R_i\}_{i=0}^{N-1}$.

Then let $\overset{N-1}{\underset{k=0}{\otimes}}\rho_k$ decomposed upon the tensor basis $\{R_i\}_{i=0}^{N-1}$, we can get $\sum_{j=0}^{N-1} x_j R_j$. Consequently the corresponding quantum discrete Fourier transform is

$$\sum_{j=0}^{N-1} x_j R_j \to \sum_{k=0}^{N-1} y_k R_k \quad , \tag{5}$$

here

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi jk/N} \quad ,$$

and $\sum_{k=0}^{N-1} y_k R_k$ is the outcome of the transform. With this quantum FD transform, the quantum linear filter that is able to separate quantum signals which is mixed in TD but independent in FD can be possibly realized.

3. Least-Squares Estimation of Quantum Signal

In the field of CSP, there is a problem of the least-squares (LS) estimation of Wiener filter, which can estimate the signal that needs detection. None the less, in the field of QSP the same problem still exists: how to subtract the indispensable signal we need from the input series. As respects, we'll give an optimal estimation of the quantum signal in the LS sense below.

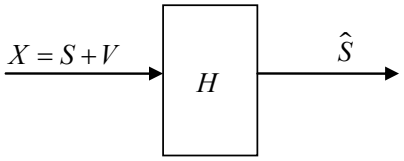

Fig. 2. Optimal LS estimation of quantum signal.

As is shown in Fig. 2, assume

$$X = S + V$$

is the input quantum signal, and

$$S = \overset{N-1}{\underset{k=0}{\otimes}}\rho_k \ , \ V = \overset{N-1}{\underset{l=0}{\otimes}}\sigma_l \ ,$$

here ρ_k and σ_l stand for the density matrices of one qubit. We hope that the quantum signal become $\hat{S}$ after it goes through the filter H, and its element $\hat{S}_n$ is the correct estimation of S_n on the criterion of LS. It means to minimize

$$e_n = \hat{S}_n - S_n$$

as much as possible. Note that $\hat{S}_n$, S_n are the nth elements of tensor $\hat{S}$ and S respectively, and

$$n = 2^N - 1 \ .$$

Therefore it's an evident result that

$$E\left[e_n^2\right] = \min \ .$$

By the principle of orthogonal projection, it means that entire e_n are orthogonal with

$$\hat{S}_n = \left(HX\right)_n \ ,$$

and

$$\left(HX\right)_n = \sum_{i=0}^{2^N-1} h_{2^{2N}-2^N+i} X_i \quad , \tag{6}$$

where $\left(HX\right)_n$ is the linear combination of X_i. So entire e_n must be orthogonal with all the X_i, in which case we have

$$E\left[e_n X_i\right] = 0, \ i = 0,1,\cdots,2^N - 1 \ . \tag{7}$$

According to equation (3.1), there are a group of equations

$$\begin{gathered} \sum_{i=0}^{2^N-1} h_{2^{2N}-2^N+i} R_{XX}\left(\left|i-j\right|\right) = R_{SX}(n-j), \\ j = 0,1,\cdots,2^N - 1, \end{gathered} \tag{8}$$

where $R_{XX}\left(\left|i-j\right|\right)$ is the autocorrelated function of X, and $R_{SX}(n-j)$ is the cross-correlated function for S and X.

In this way can we get the last 2^N elements of H. And repeating this step, the all 2^{2N} elements of H can be obtained.

Obviously the matrix H we obtained using this method complies to the axiomatic standards for quantum operations [5].

1) Let H normalized, then $0 \le tr\left[H(\rho)\right] \le 1$ for any ρ.

2) It's easy to prove that for any probability $\{p_i\}$, by the linear property of tensor H,

$$H\left(\sum_i \rho_i p_i\right) = \sum_i p_i H(\rho_i).$$

3) H is completely positive-definite.

Due to the basic principle of quantum operations, either H is a unitary operator itself, or the joint operator is unitary after the additional expansion on the H system. Therefore, the quantum transform of LS estimation can be physically realized.

Using the result of quantum LS estimation, it's easy to find that, when V is the quantum noise of a system as long as the information for autocorrelation and cross-correlation be available, we will be able to construct the optimal quantum filter in the LS sense so as to resist the quantum noise pollution. And this method of QSP provides us with a distinct approach to deal with the quantum noise in contrast with the quantum error-correcting code, which is vital for the realization of quantum computer and reliable quantum communication system.

It should be mentioned that in terms of single qubit, its probability amplitude which is a complex-valued number contains its entire information. So LS criterion can be applied to the estimation of probability amplitude. But the linear correlated function is supposed to reveal the part of the entire information for the qubit's probability amplitude. Therefore high-level correlation should be more significant on the estimation of complex-valued signals than the real-valued. Meanwhile, in terms of QSP, introducing high-level correlated function may possibly improve the accuracy of the quantum signal estimation.

4. The Optimum Detection for Non-Orthogonal State Signal

4.1. Use Least-Square Filter to Estimate Unknown Non-Orthogonal State Signals

We took the binary non-orthogonal signal as the sample, the OOK (On-Off Keying) signal is referred here generally.

$$\left.\begin{array}{ll} H_0: & |0>\\ H_1: & |\alpha> \end{array}\right\}$$

H_0 and H_1 denoting absence and presence of signal. They take the state $|0>$ and the state $|\alpha>$ as the signal carrier in turn. Then the POVM operator for OOK signal can be described as follows [10]:

$$E_1 = \frac{\left(|\alpha> - e^{-(1/2)|\alpha|^2}|0>\right)\left(<\alpha| - <0|e^{-(1/2)|\alpha|^2}\right)}{\left(1+e^{-(1/2)|\alpha|^2}\right)\left(1-e^{-|\alpha|^2}\right)},$$

$$E_2 = \frac{\left(|0> - e^{-(1/2)|\alpha|^2}|\alpha>\right)\left(<0| - <\alpha|e^{-(1/2)|\alpha|^2}\right)}{\left(1+e^{-(1/2)|\alpha|^2}\right)\left(1-e^{-|\alpha|^2}\right)},$$

$$E_3 = I - E_1 - E_2.$$

After the measurement, if the consequences turn out to be E_1, then the source signal is confirmed to be H_1, if it turn out to be E_2, the source should be H_0, and if the consequences is as E_3, the origin state of the signal is unknown. For the case of E_3 as the measurement result, the origin state of the signal will be affirmed by the least-square estimation.

Simultaneously, the error probability (the measurement outcome is E_3) is

$$P_e = e^{-1/2|\alpha|^2}, \tag{9}$$

where $|\alpha|^2$ stands for the average energy of signals.

Let L bits signal sequence passing by the POVM, assume m bits get the correct outcome, and L-m elements get the false results. We can give the estimation of the L-m elements by the m correct bits using least-square filtering.

The filter is designed with FIR (Finite Impulse Response), its impulse response vector is

$$\mathbf{h} = [h(0), h(1), \cdots, h(m-1)]^T.$$

Let

$$\mathbf{P} = \mathrm{E}\left[\mathbf{X}s_n\right],\ \mathbf{R} = \mathrm{E}\left[\mathbf{X}\mathbf{X}^T\right],$$

s_n is the unknown signal that is to be estimated,

$$\mathbf{X} = \left[s_{i_1}, s_{i_2}, \cdots, s_{i_m}\right]^T$$

plays for a vector, whose elements are m known signals, $\mathbf{R}$ is the auto-correlation function of the source, which is an $m*m$ matrix.

According to equation

$$\mathbf{P} = \mathbf{R}\mathbf{h},\ \mathbf{h} = \mathbf{R}^{-1}\mathbf{P},$$

so the unknown signal estimation is

$$\widehat{s_n} = \mathbf{h}^T\mathbf{X},$$

and its least-square error is

$$\varepsilon_{\min}(n) = \mathrm{E}\left[s_n^2\right] - \mathbf{P}^T\mathbf{R}^{-1}\mathbf{P}.$$

4.2. The Bound of Bit Error Probability for Combining POVM and Least-Square Estimation (CPLSE)

The estimation error is primarily decided by the auto-correlation function of the signal sequence and the position of the elements which to be estimated in the sequence. Concerning the auto-correlation function of the general signal is an attenuation function, we can consequently give the upper and lower bound of the estimation error.

For this instance, let's assume

$$s_1, s_2, \cdots, s_m, s_{m+1}, \cdots, s_L$$

be a signal sequence, s_1 to s_m are the known signals, s_{m+1} to s_L are the unknown signals. Of all the estimation consequences, s_L has a maximum value of the least-square error. And in the signal sequence

$$s_1, \cdots s_b, s_{b+1}, \cdots s_{b+m/2}, s_{b+m/2+1}, \\ s_{b+m/2+2}, \cdots, s_{b+m+1}, \cdots s_L$$

where s_{b+1} to $s_{b+m/2}$ and $s_{b+m/2+2}$ to s_{b+m+1} signifies the known signal, the $s_{b+m/2+1}$ signifies the midst unknown signal, and the rests symbolize the remaining unknown signals. Signal $s_{b+m/2+1}$ therefore will have the minimum value of least-square error.

Obviously, when the signal to be estimated lies in the most distant place with the known signal, it has the maximum value of least-square estimation error, while the one stands the midst of the known signals has the minimum value of estimation error, withstanding that the known signals are all adjacent with each other.

Next we'll derive the maximum and minimum bit error probability. As the bit error probability can be derived from least-square error approximately, we have:

$$\varepsilon_{\min}(n) = \sum_{s_n - \widehat{s_n}} \left(s_n - \widehat{s_n}\right)^2 P_{s_n - \widehat{s_n}},$$

s_n is the real value of the signal, $\widehat{s_n}$ is the estimation value, $P_{s_n-\widehat{s_n}}$ represents for the error probability of $s_n - \widehat{s_n}$. From equation

$$\varepsilon_{\min}(n) \approx (0.25+1)/2 \cdot P_{\left|s_n-\widehat{s_n}\right|>0.5} + (0+0.25)/2 \cdot P_{\left|s_n-\widehat{s_n}\right|<0.5} \\ = 0.625 \cdot P_{\left|s_n-\widehat{s_n}\right|>0.5} + 0.125 \cdot P_{\left|s_n-\widehat{s_n}\right|<0.5},$$

we get

$$P_{\left|s_n-\widehat{s_n}\right|>0.5} \approx 2\varepsilon_{\min}(n) - 0.25,$$

taking the principle of the least-square error into consideration, the error probability is $P_{\left|s_n-\widehat{s_n}\right|>0.5}$.

Then the bit error probability is between the maximum value of least-square error and the minimum value of least-square error, that is

$$2\varepsilon_{\min_lower} - 0.25 \le P_{e_bit} \le 2\varepsilon_{\min_upper} - 0.25 \quad (10)$$

5. Numerical Results

In this section, we make numerical analysis to verify above results. Firstly, assume the average energy $|\alpha|^2$ to be 5. Secondly, the auto-correlation function of source signals is set to be

$$R(j,k) = \frac{e^{-|j-k|}+1}{4},\ j,k \in \mathbb{Z},$$

which is shown in Fig. 3. Finally, the error probability of the POVM detection, as well as the error probability of the CPLSE are calculated with (9) and (10).

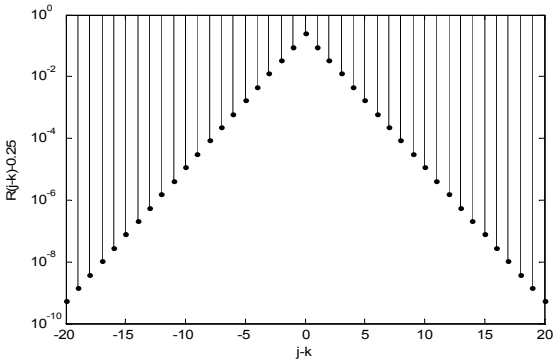

Fig. 3. Auto-correlation function of the signal sequence.

Fig. 4 is the bit error probability varying with the input signal sequence length, we set $|\alpha|^2 = 5$, L varies from 50 to 500, the outcome turn to be straight line, the POVM detection get the error probability of more than 0.08, while the CPLSE detection gets the error probability between 0.01 and 0.02.

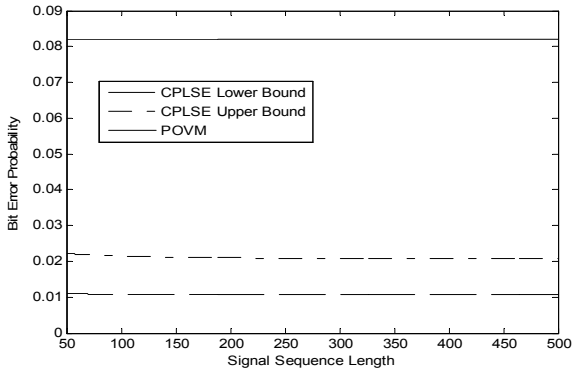

Fig. 4. Bit Error Probability of signal varying with the sequence length L.

In the case as in Fig. 5, we set $L = 100$, and $|\alpha|^2$ varies from 3 to 20. From the results we can see that with the signal power increasing, the bit error probability decrease promptly, which take on an exponential decay.

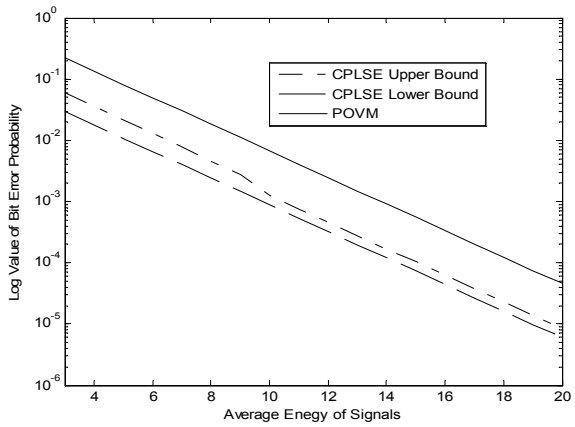

Fig. 5. Bit Error Probability of signal sequence varying with $|\alpha|^2$.

It's easy to see from all the above figures and results that the CPLSE method is able to reduce the bit error probability efficiently and thus improves the performance of coherent-state signal detection.

6. Conclusions

In this paper, we present the mathematical model of quantum signal processing. These basic works enable us to get the optimal LS estimation of quantum signal. On this basis, the POVM operators and the average bit error probability for non-orthogonal state signal are deducted. Combining the least-square error theory, we proposed the CPLSE method, which reduced the bit error probability, and improved the performance of detection. Quantum transform and the realization of quantum filter along with the quantum system analysis will be the future work.

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