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Random Sampling and Signal Reconstruction Based on Compressed Sensing

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Abstract: Compressed sensing (CS) sampling is a sampling method which is based on the signal sparse. Much information can be extracted as little as possible of the data by applying CS and this method is the idea of great theoretical and applied prospects. In the framework of compressed sensing theory, the sampling rate is no longer decided in the bandwidth of the signal, but it depends on the structure and content of the information in the signal. In this paper, the signal is the sparse in the Fourier transform and random sparse sampling is advanced by programming random observation matrix for peak random base. The signal is successfully restored by the use of Bregman algorithm. The signal is described in the transform space, and a theoretical framework is established with a new signal description and processing. By making the case to ensure that the information loss, signal is sampled at much lower than the Nyquist sampling theorem requiring rate, but also the signal is completely restored in high probability. The random sampling has following advantages: alias-free, sampling frequency need not obey the Nyquist limit, and higher frequency resolution. So the random sampling can measure the signals which their frequencies component are close, and can implement the higher frequencies measurement with lower sampling frequency. *Copyright © 2014 IFSA Publishing, S. L.*

Keywords: Compressed sensing, Random sampling, Nonuniformly sampling, Sparse sampling, Signal reconstruction.

1. Introduction

Compressed sensing (CS) looks literally like a data compression means, and is indeed for entirely different considerations. Classical data compression technology, whether it is audio compression (e.g. mp3), image compression (e.g. jpeg), video compression (mpeg), or general encoding compression (zip), are both from the characteristics of the data itself, to find and remove data implicit redundancy, so as to achieve the purpose of compression. Such compression has two characteristics: first, it occurs after the data which has been collected to complete; second, which one itself requires complex algorithms to complete. In contrast, the calculation is relatively simple in the decoding process.

The asymmetry of compression and decompression is just the opposite with the demand of people. In most cases, the data acquisition and processing equipment tend to be cheap, save electricity, less computing power portable devices, such as fool camera, or recorder, or remote monitor and so on. Responsible for processing (unzip) information is often conducted on large computers instead, it has higher computing power, and often does not portable and save energy requirements. That is to say, we are using cheap, saving energy equipment to deal with complex computing tasks, and large efficient equipment for processing is relatively simple computing tasks. This contradiction in some cases even are more acute, such as the equipment in the field operation or military operations, collecting data is often exposed in the natural environment. At any time, energy supply may lose or its performance is partial loss, even in this case, the traditional data acquisition - compression - transfer - decompression mode is basically failed.

Compressed sensing is to solve such conflicts. After collecting the data, you want to compress the redundancy, and this compression process is relatively difficult. Why do not we take directly data after compression ? Such acquisition task is much lighter, but this also eliminates the trouble of compression. This is called "compressed sensing", that is the direct perception of compressed information.

The traditional Nyquist sampling theorem requires that the sampling rate is not less than twice the highest frequency of the signal. With the development of signal processing technology and the surge in the amount of data processed, this sampling method has been far from the requirements to keep up with the high-speed signal processing. In 2006, Donoho proposed compressed sensing (CS) theory [1-3], the signal has a sparse nature, its sparse features can been used in the signal sampling points which are less original signal, and these can be approximated to restore the original signal. This theory greatly promoted the process of the signal processing theory, and has broad application prospects. Currently, compressed sensing theory have a very good application in compressed image, converting analog information, bio-sensing, signal detection and classification, wireless sensor networks, data communications and geophysical data analysis and other fields [4]. If you want to collect a few data and expect to give a lot of information in a small amount of data from these "decompression", you need to ensure that: First: the amount of the collected data contain the global information of the original signal, second: there is a kind of algorithm, by which the original signal can been restored from such a small amount of data information.

In this paper, we have researched a method of random sampling of compressed sensing and signal high-resolution reconstruction. In the method, there are the advantages of non-uniform sampling without sampling frequency limit, high frequency resolution and anti-aliasing [16]. At low sampling frequency, signal spectrum is based on non-uniform sampling in Fourier transform signal spectrum, and then signal is reconstructed by separate Bregman method [17-18], and the spectrum noise is reduced, which is caused by non-uniform sampling.

2. Random Sampling and Analysis

2.1. Uniform Sampling Drawbacks

Uniformly sampled time function is the standard linear function, the sampling time is some interval distribution. The definition of the sampled signal is $x(t)$. The sampling interval is Δt, the sampling time functions is $t_n = n\Delta t$, and the sampling frequency $f_s = \frac{1}{\Delta t}$ is satisfy of the sampling theorem, they is greater than two times the maximum signal frequency value. For finite-length signal sampling discrete $x[n] = x([1:N]\Delta t)$, N is the number of sampling points, the sampling duration $T = N\Delta t$.

By Fourier transform, to analyze the sampled signal $x(t) = \sin(2\pi ft), (f = 185Hz, N = 256, f_s = 256Hz)$.

Analysis of the signal spectrum is shown in Fig. 1.

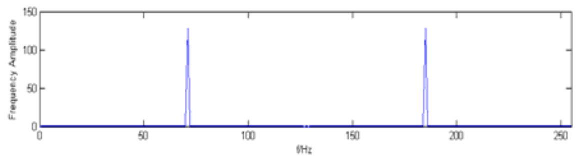

Fig. 1. Signal spectrum analysis of uniformly sampling (f_s=256 Hz).

As can be seen from Fig. 1, since the sampling frequency is less than two times of the sampling signal frequency, and it appears in the aliasing frequency 71 Hz. Because the real signal spectrum is equal with the spectrum of aliasing signal, the true signal can't be distinguished. Since this embodiment frequency resolution is 1 Hz, the signal frequency is an integer multiple of the frequency resolution, it is possible to accurately measure the frequency value.

Of the cases, the other parameters is constant, the changed sampling frequency is $f_s = 512Hz$, which is meet to the limit in sampling theorem. Analysis of the signal spectrum is shown in Fig. 2.

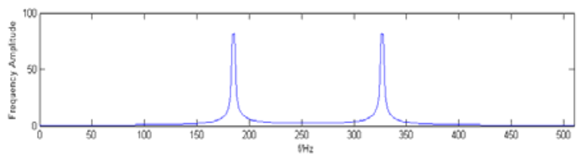

Fig. 2. Signal spectrum analysis of uniformly sampling (f_s=512 Hz).

It is showed in Fig. 2 that there are no aliasing signals in the (O,) band, but due to the change of the sampling frequency, the frequency resolution becomes 2 Hz, the true frequency is 185 Hz in the signal, and it is not an integer multiple of the frequency resolution, and thus these cause leakage of the spectrum and fences phenomenon, so the measured frequency is $f = 188\ Hz$, and it is deviated from the correct value.

As can be seen from the above analysis, uniform sampling restrictions are limited by sampling frequency; the aliasing frequency is produced; frequency resolution is not high; there are the phenomenon of spectral leakage and fences and other issues.

2.2. Random Sampling and its Fourier Transform

Random sampling is not restricted by sampling theorem, the detection range of frequencies can be achieved increasingly, and a frequency of the higher order is detected in a short length of the data and the low sampling frequency, which can quickly meet the requirements of real-time specific occasions. Most importantly, due to random sampling, so that the non-uniform sampling to eliminate the problem of aliasing frequency which is caused by uniform sampling; there is also the advantage of high frequency resolution, the spectral leakage is reduced, and the phenomenon of the fence is eliminated, etc.

In the above example, the other parameters are constant, random sampling is changed, $t_n = rand\ (0,1)T = g(n), x(n) = x(t_n)$, where rand (0,1) is random number between (0,1), $n = 1,2,\cdots,N$, g (n) is a nonlinear function of n. Fourier transform:

$$X(\omega) = \sum_{n=1}^{N} x(n) \exp(-j\omega t_n) \qquad (1)$$

The Fourier transform spectral analysis of Random samples (N = 256) is shown in Fig. 3.

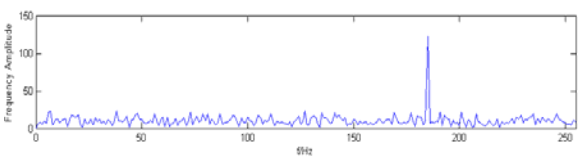

Fig. 3. Signal spectrum analysis of random sampling (Average f_s=256 Hz).

The sampling interval is increased by the random sampling time to improve the frequency resolution, and to eliminate the phenomenon of the fence. Due to random sampling, aliasing signal will not concentrate on a specific point which is related to the sampling frequency, but it is all form uniformly distributed to the signal frequency band. In addition, spectral leakage can also cause spectral noise. The noise spectrum may be reduced with increasing number of sampling points.

3. Compressed Sensing Principle

3.1. Compressed Sensing Statements

Compressive Sensing (CS) theory main idea is: Suppose that there are sparse coefficients of the signal x of length N (i.e. only a few non-zero coefficients)

of in orthogonal basis or on a tight frame Ψ. If the coefficients are projected to another observations base $\Phi : M \times N, M << N$ which is unrelated with the transform base Ψ, and a conservative collection $y{:}M \times 1$ is obtained. Then the signal x can be recorded precisely by solving an optimization problem and relying on these observations.

CS theory is a new theory framework with sampling while achieving the purpose of compression, its compressive sampling procedure is shown in Fig. 4. First, if signal $x \in R^N$ is compressible on a quadrature group or a tight frame Ψ, the transform coefficients $\alpha = \Psi^T x$ are obtained, α is the sparse representation with equivalent to x and approximation. The second step, a smooth measurement matrix Φ with $M \times N$ dimensions is designed, which is not associated with the transform base Ψ, After x is observed, x will be projected to M-dimensional space to obtain observations set $y = \Phi x$, The process is the compressing and sampling process, i. e. samples is taken [5]. Finally, the x exact or approximate approximation solution $\hat{x}$ is taken by the optimizing problem.

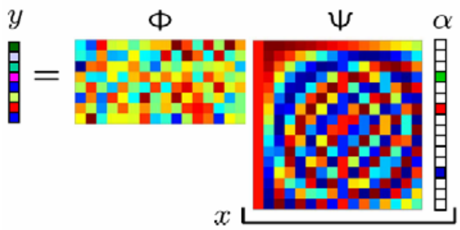

Fig. 4. Compression sampling process.

When ξ is observed with a noise,

$$y = \Phi x + \xi \qquad (2)$$

In order to be transformed:

$$\min_x \| \Psi^T x \|_1 \qquad s.t. \| y - \Phi x \|_2 < \varepsilon \qquad (3)$$

or

$$\hat{x} = \arg\min_x \frac{1}{2} \| y - \Phi x \|_2 + \lambda \| \Psi^T x \|_1 \qquad (4)$$

3.2. Separable Bregman Iterative Algorithm in Signal Recovery

Problem (4) solution is converted first into sparse vector (5) solving, when $A = \Phi\Psi$, so

$$\hat{\alpha} = \arg\min_{\alpha} \frac{\lambda}{2} \| y - A\alpha \|_2^2 + \| \alpha \|_1 \quad (5)$$

Bregman algorithm [17-20] is the following steps:

1) Calculating $B = (\lambda A^T A + I_N)^{-1}$, I_N is N-order unit matrix, $F = \lambda A^T y$; b_0, d_0 are N-dimensional zero vector;

2) Given, $\lambda(=10)$, iteration termination condition $\delta(=0.001)$, iterations $n = 1$;

3) Calculating $\alpha_n = B(F + d_{n-1} - b_{n-1})$, $d_n = sign(\alpha_n + b_{n-1})\max(|\alpha_n + b_{n-1}| - 1, 0)$, $b_n = b_{n-1} + \alpha_n - d_n$;

4) If $\| \alpha_n - \alpha_{n-1} \| \geq \delta$, $n = n + 1$, go to Step 3); Otherwise, stop the iteration, $\hat{\alpha} = \alpha_n$.

5) $\hat{x} = \Psi\hat{\alpha}$.

4. Sparse Random Sampling Design

4.1. Signal Sparse Representation

Adaptive signal transformation group ψ , the expression the signal at the base is sparse. The transformation coefficients vector of signal X is $\alpha = \psi^T X$ under transform base ψ , if 0<*P*<2 and *R*>0, these Coefficients are satisfy $\| \alpha \|_p = (\sum_i | \alpha_i |^p)^{\frac{1}{p}} \leq R$, coefficient vector is sparse. If the potential in the support domain $\{i : \alpha_i \neq 0\}$ of transform coefficients $\alpha_i = < X, \psi_i >$ is less than or equal K, $\alpha \in R^N$ is only K non-zero entries. The inherent freedom is reflected by the number K of non-zero entries of in the signal. Or the signal sparsity constitutes of a measuring scale of the number of non-zero components of the coefficients. The representation of the signal sparse can be usually measured by the vector zero-norm. A vector zero - norm is non-zero elements of this vector number. Fourier transform is what we commonly used.

4.2. Irrelevant and Isometric Nature

The adaptive and observative matrix Φ with $M \times N$ dimensions is designed which is relevant with transform base ψ . The designing objective of the observation matrix A is to restore the original signal in few observations. In a particular design, the need is consider in the following two relations:

1) The relationship between the observation matrix Φ and base matrix ψ ;

2) The relationship between the matrix $A = \Phi\psi$ and K-sparse signal α ;

First, there is irrelevance (Incoherence) between the observation matrix ϕ and the base matrix ψ . The coherence μ between the observative matrix ϕ and the base matrix ψ is defined as:

$$\mu(\Phi, \psi) = \sqrt{N} \max_{\substack{1 \leq k \leq m \\ 1 \leq j \leq N}} |< \phi_k, \psi_j >| \quad (6)$$

The coherence μ is the maximum degree of coherence of any two vectors between Φ and ψ . When the coherent vector is contained in Φ and ψ , the degree μ of the coherence is greater. In the signal compressing samples, each observation contains different information of the original signal, and the vectors between Φ and ψ are orthogonal as much as possible, the degree μ of coherence is as small as possible, and this is the incoherent reason between the basis matrix and the observation matrix [8, 12]. As long as satisfying the following equation, signal can be reconstructed in a high probability [12]

$$M \geq C\mu^2(\Phi, \Psi) K \log N \quad (7)$$

Secondly, the relationship between the matrices A and K-sparse signal is linked with the limited equidistant nature (Restricted Isometry Property, RIP) [8-11], which is the matrix "equidistant constant". To arbitrary K = l, 2... the isometric constant δ_K of the matrix A is defining as the minimum value satisfying the following formula, where α is the arbitrary K-sparse vector:

$$(1 - \delta_K) \| \alpha \|_2^2 \leq \| A\alpha \|_2^2 \leq (1 + \delta_K) \| \alpha \|_2^2 \quad (8)$$

If δ_K <1, this is unknown as which the matrix A satisfies K-RIP, it is ensured that the Euclidean distance of K-sparse signal α is constant approximately in the matrix A, which means that α is not possible in the null space of matrix A (otherwise α will have infinitely many solutions). Random matrix is commonly used to be observed in practice. A common observation matrix is Gaussian matrix, binary observation matrix, Fourier observation matrix and irrelevant observation matrix. Random observations provide an effective way to achieve compression samples.

4.3. Low-speed Rate Signal Sampling

In fact, the design of observed part is the design of efficient observation matrix, namely the useful information of sparse signal can be captured in an efficient observation (i. e., sampling) protocol which is designed, so the sparse signal is compressed into a small number of data. These agreements are non-adaptive, and only a small amount of the fixed waveform is linked with the original signal, these fixed waveforms provide concise and irrelevant signal group. Moreover, the observative process is independent of the signal itself. The reconstructed signal can be collected in the small number of observations by using optimization methods.

Sampling interval [0, T], M points are randomly collected in this interval, $t_i = rand(0,1)T, i = 1,2,\cdots,M$, $rand(0,1)$ is random point in (0,1), $x = [x(t_1), x(t_2),\cdots, x(t_M)]^T$, $y = x$, the random measurement matrix is fairly designed in random spikes base $\phi_k(t) = \delta(t-k)$, k is one of the M values which are selected randomly from [1, 2, ..., N], ψ is Fourier-base, $\psi_j(t_n) = N^{-1/2} e^{i2\pi j t_n}, j,n = 1,2,\cdots,N$, t_n belongs to the time-domain set which contains random collection and reconstructive signal. This design Φ and ψ is satisfy irrelevance (6) and restricted isometry (8). A are consisting of M (M> K) row vectors which are randomly selected from Fourier matrix A group. Namely that is a partial Fourier transform [12] method of the sampling, the Fourier transform of the signal is done, and then row vectors are randomly extracted from transform coefficients, which the observations are made with random and uncorrelated properties. The unrelated characteristics of random Observation Matrix are a sufficient condition to restore the signal properly, and the observation matrix is highly irrelevant, these can ensure the effective restoration of the signal.

4.4. Experimental Testing and Evaluation

Experimental signal function:

$$f(t) = \sin(40\pi t) + \sin(140\pi t) + \sin(300\pi t)$$

The sampling frequency $f_s = 256\ Hz$, highest signal frequency $f_{\max} = 150Hz, f_s = 256Hz < 2f_{\max} = 300Hz$. The sampling frequency is less than two times of the highest signal frequency, and it does not satisfy the Nyquist sampling theorem. In Fig. 5, there are aliasing and leakage in the Fourier transform spectrum, while random sampling is taken by using the same sampling frequency, spectrum aliasing and leakage are avoided in the discrete Fourier transformation (see Fig. 5 below). Random sampling and reconstruction are improved by the proposed compression method, and the reconstruction is very consistent in either the frequency domain or the time domain. Fig. 5 shows Result in the compressed sensing sparse sampling and the signal reconstruction with a high-resolution.

The upper picture shows the original signal and the reconstructed signal in time domain, the middle picture shows the uniform sampling of the Fourier transform, the lower part are the random sample of the original signal, the frequency-domain recovered signal, the relative error (Relative error = =0.1787) of the reconstructed signal in time domain.

5. Conclusion and Outlook

The basic idea of compressed sensing is that much information [13-22] is extracted from little as possible the data, and it is no doubt that there is a great prospect theory and application ideas. It is an extension of the traditional information theory, but it has gone beyond traditional compression theory, and it has formed a new sub-branch. CS theory states that when the signal has a sparse feature, the signal can be accurately reconstruct by a small number of observations which is much smaller than the length of the signal source. In CS theory, signal sampling and compression are combined into a single step, and the signal is encoded, these have broken the traditional limits of the Nyquist sampling theorem in some extent, and have reduced the burden on the hardware processing.

In the framework of compressed sensing theory, the sampling rate is no longer determined by the bandwidth of the signal, but it depends on the structure and content of information in the signal. The signal is described by the use of a transform space, a new theoretical framework is built to describe and process signal, so that in the circumstances to ensure that information is not lost, the sampling rate of the signal is far less than is required in the Nyquist sampling theorem, but also the signal is recovered fully in high probability.

The sampling rate of sampling system can be effectively improved by using random sampling technique as a non-uniform sampling method. In a random sample, the non-uniform distribution characteristics of the sampling interval lead to collect no enough samples for signal reconstruction. In this paper, if the signal has the sparsity in the Fourier transform, the random observative matrix is designed in the spikes random group that is the random sparse sampling. The signal is successfully restored by using Bregman iterative algorithm. The hardware cost is not increased in this method, and the original signal is reconstructed by the limited random sample values. Experiments show that sparse signal sampling in the frequency domain is far below the Nyquist frequency signal sampling rate, and the original signal can been accurately reconstruct by compressed sensing signal reconstruction algorithm.

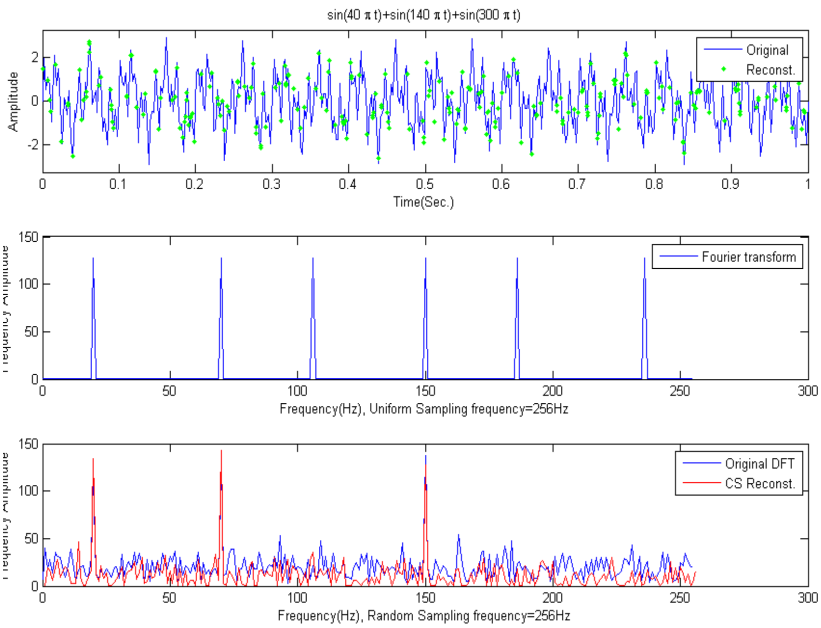

Fig. 5. Compressed sensing sparse sampling and signal reconstruction.

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