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# On the Steady-State Behaviour of Discrete-Time Linear Systems with Poles on the Unit Circle

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**Abstract:** The steady-state behaviour of discrete-time systems defined by difference equations whose transfer functions may have poles on the unit circle is studied. It is shown that, contrarily to the regular cases, the eigenfunctions of these systems are no longer the exponentials, but, if the input function is a product of a falling factorial by an exponential, the output is a linear combination of such kind of functions. The very useful and well-known ARIMA case is studied and exemplified. *Copyright © 2016 IFSA Publishing, S. L.*

**Keywords:** Ordinary difference equations, Constant coefficient, Particular solution, Eigenfunction, Transfer function, Singular difference equation.

## 1. Introduction

The systems defined by constant coefficient ordinary difference equations have a long tradition in applied sciences and have a large amount of engineering applications, mainly in Signal Processing [1, 3-4, 11] where they are referred as ARMA (Autoregressive Moving Average) models. In these fields the difference equations are written in the general format

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k), \qquad (1)$$

where $n \in Z$, $M, N \in Z_0^+$ and the coefficients $a_k$, $k = 0, 1, ..., N$ and $b_k$, $k = 0, 1, \ldots, M$ are real constants. We could consider also fractional delays as in [5], but we will not do it here. In the regular case the response of these systems to a sinusoid is also a sinusoid with the same frequency [10-11] which leads to introduce the frequency response that is another way of describing the system, being equivalent to (1). In a general formulation we can say that the exponentials, $\beta^n$, $n \in \mathbf{Z}$ and $\beta \in \mathbf{C}$, are the eigenfunctions of these systems.

The situation is not so simple in the singular case that we will study in this paper. However and as we will show, the role of the exponentials is played by functions defined as the product of a falling factorial by an exponential [8].

Let $(n)_k = n(n-1)(n-2)...(n-k+1)$, if $k \geq 1$, and $(n)_0=1$, be the Pochhammer symbol for the falling factorial. We will assume that the input, x(n), is the product of a falling factorial and an exponential defined on Z:

$$x(n) = (n)_K \beta^n, \qquad (2)$$

where β is any complex number and K a positive integer. This function does not have either Z transform or Fourier transform [11]. As we will show, this kind of functions are not eigenfunctions of the system (1), but we can affirm:

*When the input of the system is a function of the type (2) the output is a linear combination of several similar functions*.

This statement is valid for any regular or irregular system, although we will pay a special attention to the irregular cases, mainly the Autoregressive Integrated Moving Average (ARIMA) models. Therefore, the frequency responses of these systems loose the normal interpretation. This problem was never considered with generality.

The procedure presented here is formally similar to the one followed in [6-7, 9] for dealing with systems defined by differential equations. We will start by introducing the eigenfunctions of difference equations and compute the corresponding eigenvalues. These are used to obtain the particular solutions we are looking for. Several examples are presented to illustrate the behaviour of the approach.

The objective of this paper is the study of singular cases corresponding to the situations where the transfer function becomes infinite; such situations are treated with all the generality [8]. The important ARIMA model is a particular case with a pole at 1. We will show how to compute the output for these cases.

The paper outline is as follows. In Section 2, we will introduce the exponentials as eigenfunctions of the ARMA systems. The generalization for the input as in (2) is done in Section 3. The singular cases are treated in Section 4 where the particular ARIMA. At last, we will present some conclusions.

## 2. The Exponentials as Eigenfunctions

The discrete convolution is defined by:

$$x(n) * y(n) = \sum_{k=-\infty}^{+\infty} x(k)y(n-k), n \in \mathrm{Z} \qquad (3)$$

This operation has several interesting properties, but we will study only those interesting for the development we intend to do.

1) Let the Kronecker delta be defined by

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \qquad (4)$$

As it is easy to verify, this function is the neutral element of the convolution

$$x(n) = \delta(n) \times x(n)$$

2) The convolution is commutative

In fact

$$x(n) \times y(n) = y(n) \times x(n)$$

as it is easily verified with the substitution $m = n-k$ in (3).

3) A shift in one factor produces the same shift in the convolution. Let $z(n) = x(n) \times y(n)$. Then

$$x(n - n_0) \times y(n) = z(n - n_0)$$

and using the commutability

$$y(n - n_0) \times x(n) = x(n - n_0) \times y(n)$$

For proof we start from (3)

$$x(n-n_0) \times y(n) = \sum_{k=-\infty}^{+\infty} x(k-n_0)y(n-k), n \in \mathrm{Z}$$

and substitute m for $k - n_0$ to get

$$x(n-n_0) \times y(n) = \sum_{k=-\infty}^{+\infty} x(m)y(n-n_0-m), n \in \mathrm{Z}$$

With these properties at hand we return to our objective of computing the eigenfunction for Equation (1). Consider a particular input $x(n) = \delta(n)$ and let the corresponding solution be h(n) that we will call Impulse Response. So, this is the solution of

$$\sum_{k=0}^{N} a_k h(n - k) = \sum_{k=0}^{M} b_k \delta(n - k) \qquad (5)$$

Now convolve both sides in (5) with x(n).

$$\sum_{k=0}^{N} a_k h(n - k) \times x(n) = \sum_{k=0}^{M} b_k \delta(n - k) \times x(n)$$

Using the above properties of the convolution we can write

$$\sum_{k=0}^{N} a_k [h(n - k) \times x(n)] = \sum_{k=0}^{M} b_k x(n - k)$$

A comparison of this equation with (1) allows us to conclude that its solution is given by

$$y(n) = x(n) \times h(n) \qquad (6)$$

This means that the solution of (1) is the convolution of x(n) with the impulse response.

*Theorem 2.1:* – The particular solution of the difference equation (1) when $x(n) = z^n$, $z \in \mathrm{C}$, $n \in \mathrm{Z}$ is given by

$$y(n) = H(z) \cdot z^n \qquad (7)$$

provided that H(z) exists.

*Proof:* Insert $x(n) = z^n$ into (6) and use (3) to get

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)z^{n-k} \quad = \sum_{k=-\infty}^{\infty} h(k)z^{-k}z^n$$

With

$$H(z) = \sum_{k=-\infty}^{\infty} h(k)z^{-k} \qquad (8)$$

we obtain (7). H(z) is called *Transfer Function* of the system defined by the difference Equation (1) and as seen is the Z transform of the impulse response. We can give it another more interesting format by inserting (7) into (1) to obtain immediately

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \qquad (9)$$

This theorem shows that the exponentials are the eigenfunctions of the constant coefficient ordinary difference equations, provided that H(z) is finite for the particular value of z at hand.

In the following we will consider that the *characteristic polynomial* in the denominator is not zero for such particular value of z. Later we will consider the cases where z is a zero of the characteristic polynomial – z is a pole of the system.

**Example 1**

Let $x(n) = 2^n$ and consider the equation

$$y(n) = x(n) + x(n-1)$$

We have $H(z) = 1 + z^{-1}$. So, the particular solution is given by

$$y(n) = H(2)\cdot 2^n = \frac{3}{2}2^n$$

Let now $x(n) = (-1)^n$. We have

$$y(n) = H(-1)(-1)^n \equiv 0$$

Therefore, the oscillating exponential is eliminated: this is why we use the term *filters* when we refer to linear systems.

**Example 2**

Consider the difference equation

$$y(n)+y(n-1)-4y(n-2)+2y(n-3) = x(n)+2x(n-1)$$

Let $x(n) = (1/2)^n$. The solution is given by:

$$y(n) = \frac{1+2(\frac{1}{2})^{-1}}{1+(\frac{1}{2})^{-1}-4(\frac{1}{2})^{-2}+2(\frac{1}{2})^{-3}}\cdot\left(\frac{1}{2}\right)^n = \frac{5}{3}\cdot\left(\frac{1}{2}\right)^n$$

showing that $(1/2)^n$ is really an eigenfunction of the system with eigenvalue equal 5/3.

*The sinusoidal case*

In a particular setting, put $z = e^{i\omega_0}$. We obtain immediately

$$y(n) = H(e^{i\omega_0})\cdot e^{i\omega_0 n} \qquad (10)$$

**Example 3**

Consider the difference equation

$$y(n) + y(n-1) - 4y(n-2) + y(n-3) = x(n) + 2x(n-1)$$

Let $(n) = e^{i\frac{\pi}{2}n}$. The solution is given by:

$$y(n) = \frac{1}{1+i^{-1}-4i^{-2}+2i^{-3}}\cdot e^{i\frac{\pi}{2}n} = \frac{1}{5}\cdot e^{i\frac{\pi}{2}n}$$

The above result, (10), is very interesting since it allows us to compute easily the solution when $x(n) = \cos(\omega_0 n)$ or $x(n) = \sin(\omega_0 n)$. Consider the first case; the second is similar. We have

$$x(n) = \cos\omega_0 n = \frac{1}{2}\cdot e^{i\omega_0 n} + \frac{1}{2}\cdot e^{-i\omega_0 n}$$

that leads to

$$y(n) = \frac{1}{2}H(e^{i\omega_0})e^{i\omega_0 n} + \frac{1}{2}H(e^{-i\omega_0})e^{-i\omega_0 n}$$

The function

$$H(e^{i\omega}) = |\,H(e^{i\omega})\,|e^{i\varphi(e^{i\omega})} \qquad (11)$$

is called *frequency response* of the system in engineering applications.

It is not very hard to show that if the coefficients in (1) are real, the *amplitude spectrum*, $|\,H(e^{i\omega})\,|$, is an even function, while the *phase spectrum*, $\varphi(e^{i\omega})$, is an odd function.

These properties allow us to state the following result.

*Theorem 2.2:* – The particular solution of the difference equation (1) when $x(n) = \cos\omega_0 n$ is given by

$$y(n) = \left|H(e^{i\omega_0})\right|\cos[\omega_0 n + \varphi(e^{i\omega_0})] \qquad (12)$$

*Proof:* According to what we said above,

$$\left|H(e^{i\omega_0})\right| = \left|H(e^{-i\omega_0})\right|$$

and

$$\varphi(e^{i\omega_0}) = -\varphi(e^{-i\omega_0})$$

which leads to

$$y(n) = \left| H(e^{i\omega_0}) \right| \frac{1}{2}\left[e^{i\omega_0 n} e^{i\varphi(e^{i\omega_0})} + e^{-i\omega_0 n} e^{-i\varphi(e^{i\omega_0})}\right]$$

giving immediately to the result.

It is important to remark that when $H(e^{i\omega_0}) = 0$, y(n) is identically null. According to we said above, the system filters out the $e^{i\omega_0 n}$ component. This theorem states clearly the importance of the frequency response of a system.

**Example 4**

Consider again the equation of Example 3, but with a change in the right hand side:

$$y(n) + y(n-1) - 4y(n-2) + y(n-3) = 3x(n) - 4x(n-1)$$

and assume that $x(n) = sin\ \frac{\pi}{2}\ n$. Then

$$H(z) = \frac{3 - 4\, z^{-1}}{1 + z^{-1} - 4z^{-2} + z^{-3}}$$

and

$$y(n) = \frac{1}{2i}\,\frac{3 - 4\, e^{-i\frac{\pi}{2}}}{1 + e^{-i\frac{\pi}{2}} - 4e^{-i\pi} + e^{-i\frac{3\pi}{2}}}\, e^{i\frac{\pi}{2}n} - \frac{1}{2i}\,\frac{3 - 4\, e^{i\frac{\pi}{2}}}{1 + e^{i\frac{\pi}{2}} - 4e^{i\pi} + e^{i\frac{3\pi}{2}}}\, e^{i\frac{\pi}{2}n}$$

allowing to get

$$y(n) = \sin\left(\frac{\pi}{2}n + \theta\right)$$

with $\theta = \cot^{-1}({}^4/_3)$.

## 3. Product of a Falling Factorial by an Exponential

To go further we are going to consider the case

$$x(n) = (n)_K\, \beta^n,\ n\in Z,\ K\in N_0$$

Although not so important in applications as is the previous case, it constitutes a simple generalization that is interesting for studying the complete behavior of the system.

It is not difficult to see that we can write

$$x(n) = \beta^K \lim_{z\to\beta} \frac{d^K}{dz^K}\, z^n$$

Return to (6) and particularize for our case to obtain:

$$y(n) = \sum_{k=-\infty}^{+\infty} h(k)\,(n-k)_K\, z^{n-k} = \sum_{k=-\infty}^{+\infty} h(k)\,\frac{d^K}{dz^K}\, z^{n-k}$$

For z in the region of convergence of the Z transform, the series converges uniformly and we can commute the derivative and summation operations. This procedure leads to the next theorem.

*Theorem 3.1:* – The particular solution of the difference equation (1) when $x(n) = (n)_K\, \beta^n$, is given by

$$y(n) = \beta^K \lim_{z\to\beta} \frac{d^K[H(z)z^n]}{dz^K} \qquad (13)$$

Using the Leibniz rule we can obtain another expression for y(n) stated in as follows.

*Corollary 3.1:* -- The particular solution of the difference Equation (1) when $x(n) = (n)_K\, \beta^n$ is given by:

$$y(n) = \sum_{j=0}^{K} \binom{K}{j}\ H^{(j)}(\beta)\,(n)_{K-j}\, \beta^n \qquad (14)$$

The special case corresponding to $x(n) = (n)_K$ leads to the output

$$y(n) = \sum_{j=0}^{K} \binom{K}{j}\ H^{(j)}(1)\,(n)_{K-j} \qquad (15)$$

**Example 5**

Consider again the equation of Example 4:

$$y(n) + y(n-1) - 4y(n-2) + y(n-3) = 3x(n) - 4x(n-1)$$

and assume that $x(n) = n$. We obtain immediately

$$y(n) = \sum_{j=0}^{1} \binom{1}{j}\ H^{(j)}(1)\,(n)_{1-j} = H(1)\cdot n + H'(1)$$

As it can be shown, H(1) = 1 and H'(1) = 0 which implies that the solution is

y(n) = n

a curious result.

## 4. The Singular Case – Arima

Consider now the situation where the *characteristic polynomial* has an m[th] order root for a given z = β value. To look for a solution, assume that

$$x(n) = w(n)\beta^n$$

and

$$y(n) = v(n)\beta^n$$

Insert x(n) and y(n) into (1) to obtain a new equation

$$\sum_{k=0}^{N} a_k \beta^{-k} v(n - k) = \sum_{k=0}^{M} b_k \beta^{-k} w(n - k) \quad (16)$$

with corresponding transfer function H(βz). In fact with the above substitutions, we moved the pole from z=β to z=1.This means that the new, (16), has a $m^{th}$ order pole at z = 1. We can say that we transformed the singular system into an *ARIMA system*. This kind of systems appears frequently in econometric studies.

In terms of the variable n we have an mth order differentiation at the output. This is equivalent to do an anti-difference on the input.

We introduce the D operator defined by the differencing operation

$$Dv(n) = v(n) - v(n - 1)$$

Let D−m represent the mth order anti-difference operator – D−m Df(n) = DD−m f(n) = f(n) – that is essentially the mth order primitive without primitivation constants. This suggests the substitution v(n) = D−mu(n) to obtain the new difference equation

$$\sum_{k=0}^{N-m} \bar{a}_k u(n - k) = \sum_{k=0}^{M} \bar{b}_k w(n - k), \quad (17)$$

where $\bar{a}_k = a_k \beta^{-k}, k = 0, 1, \cdots, N - m,$ are the coefficients of the new characteristic polynomial,

$$\bar{A}(z) = \frac{A(\beta z)}{(1 - z^{-1})^m}$$

and $\bar{b}_k = b_k \beta^{-k}, k = 0, 1, \cdots, M,$ are the coefficients of the new numerator polynomial $\bar{B}(z) = B(\beta z)$.

Now, we are interested in solving (17) for the particular case, w(n) = $(n)_K$. Therefore, we will use Relation (15), allowing to obtain the following result.

*Theorem 4.1:* – The particular solution of the difference Equation (1) when x(n) = $(n)_K \beta^n$ with A(β) = 0 is given by

$$y(n) = \beta^n D^{-m} \left[ \sum_{j=0}^{K} \binom{K}{j} \bar{H}^{(j)}(1)\, (n)_{K-j} \right], \quad (18)$$

where K is a positive integer and $\beta$ is a complex number. The new transfer function is

$$\bar{H}(z) = \frac{\bar{B}(z)}{\bar{A}(z)} = \frac{(1 - z^{-1})^m B(\beta z)}{A(\beta z)}$$

We can give (18) a new format, provided that we use the following recursively obtained relation

$$D^{-m}(n)_K = \frac{K!}{(K + m - j)!} (n)_{K+m}$$

Then we can write

$$y(n) = \beta^n D^{-m} \left[ \sum_{j=0}^{K} \binom{K}{j} \bar{H}^{(j)}(1) \frac{(K - j)!}{(K + m - j)!} (n)_{K+m-j} \right] \quad (19)$$

If K = 0 (pure exponential input), we obtain:

$$y(n) = \beta^n \bar{H}(1) \frac{1}{m!} (n)_m \quad (20)$$

Making $\beta = e^{i\omega_0}$, we are led to conclude that the response of the ARIMA model to a pure sinusoid is never a pure sinusoid: the amplitude increases with time. This is the reason why this model is used for modeling non-stationary situations.

**Example 6**

Consider the equation

$$y(n) - y(n - 1) - 4y(n - 2) - 2y(n - 3) = x(n)$$

with x(n) = $n(-1)^n$. The point z = −1 is a pole of the transfer function, A(−1) = 0, of order m = 1. On the other hand,

$$\bar{H}(z) = \frac{1 + z^{-1}}{1 - z^{-1} - 4z^{-2} - 2z^{-3}} = \frac{1}{1 - 2z^{-1} - 2z^{-2}}$$

and

$$\overline{H'}(z) = \frac{2z^{-2} + 4z^{-3}}{(1 - 2z^{-1} - 2z^{-2})^2}$$

leading to $\bar{H}(-1) = 1$ and $\bar{H}'(-1) = 2$. Therefore, the solution is

$$y(n) = \left[\frac{1}{2}(n)_2 + 2n\right](-1)^n = \frac{1}{2} n(n + 3)(-1)^n$$

**Example 7**

To exemplify the ARIMA case, take the following equation with x(n) = 1

$$y(n) - 2y(n-1) + 3y(n-2) - 2y(n-3) = x(n)$$

The point z = 1 is a pole of the transfer function, A(1) = 0, of order m = 1. On the other hand,

$$\overline{H}(z) = \frac{1}{1 - z^{-1} + 2z^{-2}}$$

giving $\overline{H}(-1) = 1/2$. The solution is

$$y(n) = n/2$$

**Example 8**

The oscillator is a very interesting system that can be defined by the equation

$$y(n) - 2\cos(\omega_0)y(n-1) + y(n-2) = x(n) - \cos(\omega_0)x(n-1)$$

Now, let $x(n) = e^{i\omega_0 n}$. The system has two simple (m = 1) poles at $e^{\pm i\omega_0 n}$. Proceeding as described above, we obtain $\overline{H}(e^{i\omega_0}) = 1/2$ and the output is easily computed

$$y(n) = \frac{1}{2} n\, e^{i\omega_0 n}$$

As we said above, it is a non-stationary model.

## 5. Conclusions

The singular steady-state output in discrete-time linear systems was studied taking as base the similar study made for the regular case. In this an eigenfunction approach was used. For the singular case, products of falling factorials and exponentials were used to play the role of the exponentials. Some examples were used to illustrate the procedure, in particular the ARIMA case was considered. This formulation can be used to study the autocorrelation function of the output when the input is a stationary stochastic process.

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