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Semi-Implicit Additive Operator Splitting Scheme for Image Segmentation Using the Chan-Vese Model

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Abstract: Active contour models are designed to evolve an initial curve, called level set, to extract the desired object(s) in an image. Most approaches are based on semi-implicit schemes which are stable for all time steps. Various models are used for the global segmentation such as Chan-Vese (CV) model. The CV model has the global segmentation property to segment all objects in an image. The problem with this model is the high time computing. In order to reduce it, our contribution in this work is the association of a semi-implicit Additive Operator Splitting (AOS) technique with the CV model in biphase and multiphase cases. The basic idea behind AOS schemes is to decompose a multi-dimensional problem into one-dimensional ones that can be solved very efficiently. In this paper, we present the new association in biphase and multiphase cases with simulations showing the efficiency of the proposed method.

Keywords: Image segmentation, Active contours, Chan Vese, AOS scheme, Level set.

1. Introduction

Image segmentation is the task of partitioning an image into multiple regions. The most known region based method has been proposed by Mumford Shah [1] who have introduced a general optimization framework. To determine desired curves or surfaces, this method uses an energy functional based on regional geometric properties such as the area of the region, its contour length and the variation of individual pixel intensities inside and outside the region. However, the Mumford Shah [2] model cannot be easily implemented. The CV method [2] is a special implementation of Mumford Shah using a level set function for the case of two phases with two piecewise constants. The basic idea of CV model is to minimize energy functional by solving the Euler-Lagrange equation. This minimization takes enough time in image segmentation.

To reduce the time of segmentation, Weickert et al. [3] provide a fast algorithm using the semi-implicit AOS scheme. The basic idea behind the AOS schemes is to decompose a multi-dimensional problem into one-dimensional ones that can be solved very efficiently. Then the final multi-dimensional solution is approximated by averaging the one-dimensional solutions. In [4], the authors present a combination of the semi-implicit AOS scheme and a narrow-band technique which is associated to the geodesic active contours. This association requires re-initialization for each iteration which is the weakness of the method. As solution, Kuhne et al. [5] provide a fast algorithm using a semi-implicit AOS scheme technique which is suitable both for the geometric and the geodesic active contour model. In [6], the authors propose a new selective segmentation model, combining ideas from global segmentation that can be reformulated in a convex way such that a global minimizer can be found independently of initialization. They present the Convex Distance Selective Segmentation (CDSS) functional (based on CV model) which is associated with the semi-implicit AOS scheme. In our work, we use a level set representation of the CV model with the semi-implicit AOS scheme in order to improve the speed of the segmentation in biphase and multiphase cases.

This paper is organized as follows. Section 2 contains a review of level set method and the CV model for biphase and multiphase cases. In Section 3, we present the semi-implicit AOS scheme. Then, we present the CV model with the semi-implicit AOS scheme in biphase and multiphase cases in Section 4. Experimental results are given in Section 5.

2. Active Contour Models

In this section, we shall first provide an overview of level set theory before we get into the details of the CV model.

2.1. Level Set Method

A level set method is a numerical technique, which helps with tracking moving fronts to interfaces and shapes. This technique was first introduced by Osher et al. in [7], where the boundaries are given by level sets of a function $\varphi(\mathrm{x})$, naming it as the level set method. This method is very successful due to a very easy way of following shapes that change topology. For a given interface $\Gamma = \partial\Omega$ as shown in **Fig. 1**, the level set is independent of the parametrisation of the contour and can be used to represent the interface evolution. The idea of the level set method is to implicitly represent an interface Γ as the level set of a function φ. The level set function φ of the closed front Γ is defined as follows:

$$\begin{cases} \varphi(\mathrm{x}) > 0 & \text{inside } \Gamma \\ \varphi(\mathrm{x}) < 0 & \text{outside } \Gamma, \\ \varphi(\mathrm{x}) = 0 & \text{on } \Gamma. \end{cases}$$

where $\mathrm{x} \in R^2$.

The adjusting contour at time t is denoted by $\varphi(\mathrm{x}(t)\ ;t)$

$$\begin{cases} \varphi(\mathrm{x}(t);t) > 0 & \text{inside } \Gamma \\ \varphi(\mathrm{x}(t);t) < 0 & \text{outside } \Gamma, \\ \varphi(\mathrm{x}(t);t) = 0 & \text{on } \Gamma. \end{cases}$$

The level set value of a point on the contour with motion must always be 0.

$$\varphi(\mathrm{x}(t);t) = 0, \tag{1}$$

A derivation of (1) with respect to t and after some manipulation, yields PDE equation:

$$\frac{\partial \varphi}{\partial t} + F\left|\nabla \varphi\right| = 0\,, \tag{2}$$

where F stands for the speed in which the contour propa-gates in normal direction with an initial condition $\varphi(\mathrm{x}, t=0)$ (the initial drawn curve)

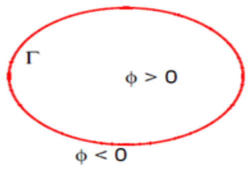

Fig. 1. Representation of the interface Γ.

2.2. The Chan-Vese Model

2.2.1. Biphase Case

In [2], the authors present a special implementation of the CV method based on the use of the level set method to minimize the piecewise constant two phases Mumford Shah functional [1]. The advantage of this implementation is the possibility to detect objects whose boundaries are not necessarily defined by gradient and overcame the problematic tracking of Γ. For a given image u_0 in domain Ω, the CV model is formulated by minimizing the following energy functional:

$$\begin{aligned} F^{CV} = \mu \int_\Omega \delta(\varphi)\left|\nabla\varphi\right| dxdy + \nu \int_\Omega H(\varphi)dxdy \ + \\ \lambda_1 \int_\Omega \left|u_0(x,y) - c_1\right|^2 H(\varphi(x,y))dxdy + \\ \lambda_2 \int_\Omega \left|u_0(x,y) - c_2\right|^2 (1 - H(\varphi(x,y)))dxdy \end{aligned} \tag{3}$$

where μ, λ_1 and λ_2 are positive parameters, φ is a level set function, $H(\varphi)$ is the Heaviside function and $\delta(\varphi)$ is the Dirac function. Generally, the regularized versions are selected as follows:

$$\begin{cases} H_\varepsilon(\varphi) = \dfrac{1}{2}\left(1 + \dfrac{2}{\pi}\ \arctan\left(\dfrac{\varphi}{\varepsilon}\right)\right) \\ \delta_\varepsilon(\varphi) = \dfrac{1}{\pi}\dfrac{\varepsilon}{\varphi^2 + \varepsilon^2} \end{cases}, \tag{4}$$

The two piecewise constants c_1 and c_2 are defined as

$$c_1 = \frac{\int_\Omega u_0(x,y) H_\varepsilon(\varphi(x,y))dxdy}{\int_\Omega H_\varepsilon(\varphi(x,y))dxdy}, \tag{5}$$

$$c_2 = \frac{\int_\Omega u_0(x,y)\left(1-H_\varepsilon(\varphi(x,y))\right)dxdy}{\int_\Omega\left(1-H_\varepsilon(\varphi(x,y))\right)}, \quad (6)$$

The evolution equation is given by:

$$\frac{\partial\varphi}{\partial t} = \delta_\varepsilon(\varphi)\begin{bmatrix}\mu\,\nabla\cdot\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right)-\nu-\lambda_1(u_0-c_1)^2 \\ +\lambda_2(u_0-c_2)^2\end{bmatrix}, \quad (7)$$

2.2.2. Multiphase Case

The CV model for multiphase piece-wise constant (we use two level set functions φ_1 and φ_2) is formulated by minimizing the following energy functional [8]:

$$\begin{aligned}F = {} & \mu\int_\Omega\left|\nabla H_\varepsilon(\varphi_1)\right|dxdy+\mu\int_\Omega\left|\nabla H_\varepsilon(\varphi_2)\right|dxdy+ \\ & \int_\Omega(u_0-c_{11})^2H_\varepsilon(\varphi_1)H_\varepsilon(\varphi_2)dxdy+ \\ & \int_\Omega(u_0-c_{10})^2H_\varepsilon(\varphi_1)\left(1-H_\varepsilon(\varphi_2)\right)dxdy+ \\ & \int_\Omega(u_0-c_{01})^2\left(1-H_\varepsilon(\varphi_1)\right)H_\varepsilon(\varphi_2)dxdy+ \\ & \int_\Omega(u_0-c_{00})^2\left(1-H_\varepsilon(\varphi_1)\right)\left(1-H_\varepsilon(\varphi_2)\right)dxdy+\end{aligned} \quad (8)$$

where

$$c_{11} = \frac{\int_\Omega u_0H_\varepsilon(\varphi_1)H_\varepsilon(\varphi_2)dxdy}{\int_\Omega H_\varepsilon(\varphi_1)H_\varepsilon(\varphi_2)dxdy}, \quad (9)$$

$$c_{10} = \frac{\int_\Omega u_0H_\varepsilon(\varphi_1)\left(1-H_\varepsilon(\varphi_2)\right)dxdy}{\int_\Omega H_\varepsilon(\varphi_1)\left(1-H_\varepsilon(\varphi_2)\right)dxdy}, \quad (10)$$

$$c_{01} = \frac{\int_\Omega u_0\left(1-H_\varepsilon(\varphi_1)\right)H_\varepsilon(\varphi_2)dxdy}{\int_\Omega\left(1-H_\varepsilon(\varphi_1)\right)H_\varepsilon(\varphi_2)dxdy}, \quad (11)$$

$$c_{00} = \frac{\int_\Omega u_0\left(1-H_\varepsilon(\varphi_1)\right)\left(1-H_\varepsilon(\varphi_2)\right)dxdy}{\int_\Omega\left(1-H_\varepsilon(\varphi_1)\right)\left(1-H_\varepsilon(\varphi_2)\right)dxdy}, \quad (12)$$

Evolution equations of φ_1 and φ_2 are given by:

$$\frac{\partial\varphi_1}{\partial t} = \delta_\varepsilon(\varphi_1)\left\{\begin{matrix}\mu\ div\ \left(\frac{\nabla\varphi_1}{|\nabla\varphi_1|}\right)- \\ \begin{bmatrix}\left((u_0-c_{11})^2-(u_0-c_{01})^2\right)\left(H_\varepsilon(\varphi_2)\right)+ \\ \left((u_0-c_{10})^2-(u_0-c_{00})^2\right)\left(1-H_\varepsilon(\varphi_2)\right)\end{bmatrix}\end{matrix}\right\} \quad (13)$$

$$\frac{\partial\varphi_2}{\partial t} = \delta_\varepsilon(\varphi_2)\left\{\begin{matrix}\mu\ div\ \left(\frac{\nabla\varphi_2}{|\nabla\varphi_2|}\right)- \\ \begin{bmatrix}\left((u_0-c_{11})^2-(u_0-c_{10})^2\right)\left(H_\varepsilon(\varphi_1)\right)+ \\ \left((u_0-c_{01})^2-(u_0-c_{00})^2\right)\left(1-H_\varepsilon(\varphi_1)\right)\end{bmatrix}\end{matrix}\right\} \quad (14)$$

3. Additive Operator Splitting Scheme

The AOS method is proposed by Tai et al. in [9] and Weickert et al. in [3]. The AOS scheme guarantees equal treatment of all coordinate axes and is stable for big time steps. The scheme presents the semi-implicit algorithm based on a discrete non-linear diffusion scale-space framework. This scheme is applied to the m-dimensional diffusion equation and it is given in the following form:

$$\frac{\partial\varphi}{\partial t} = div(g\nabla\varphi)+f(\mathrm{x},\varphi), \quad (15)$$

$$\frac{\partial\varphi}{\partial t} = \sum_{j=1}^{m}\frac{\partial}{\partial x_j}\left(g_j(\varphi)\frac{\partial\varphi}{\partial x_j}\right)+f(\mathrm{x},\varphi), \quad (16)$$

where $[0,T]\times\Omega\subset R^m$. The initial and boundary conditions are:

$$\varphi(0,.)=\varphi_0 \text{ and } \frac{\partial\varphi}{\partial \mathrm{n}}=0 \text{ on } \partial\Omega,$$

We consider discrete times $t_k = k\Delta t$, where $k\in N_0$ and Δt a semi-implicit discretization of the diffusion equation.

$$\varphi^{k+1} = \left(I-\Delta t\sum_{l=1}^{m}A_l(\varphi)\right)^{-1}\hat{\varphi}^k, k=1,2,... \quad (17)$$

where $\hat{\varphi}^{\,k} = \varphi^k+\Delta tf$.

We may consider AOS variant (for $m=2$)

$$\varphi^{k+1} = \frac{1}{2}\sum_{l=1}^{2}\left(I-2\Delta tA_l\left(\varphi^k\right)\right)^{-1}\hat{\varphi}^k, k=1,2... \quad (18)$$

The AOS scheme offers one important advantage [10]: the operators $B_l\left(u^k\right)=I-2\Delta tA_l\left(\varphi^k\right)$ lead to strictly diagonally dominant tridiagonal linear systems, which can be solved very efficiently with Thomas algorithm. This algorithm has a linear complexity and can be implemented very easily.

To implement equation (18), we proceed in three steps [10]:

1) Evolution in x direction with step size $2\Delta t$:

Solve the tridiagonal system $\left(I-2\Delta tA_x\left(\varphi^k\right)\right)\upsilon^{k+1}=\hat{\varphi}^k$ for υ^{k+1} .

2) Evolution in y direction with step size $2\Delta t$:

Solve the tridiagonal system $\left(I-2\Delta t A_y\left(\varphi^k\right)\right)\omega^{k+1}=\hat{\varphi}^k$ for ω^{k+1}.

3) Averaging:

Compute $\varphi^{k+1} := 0.5\left(\upsilon^{k+1}+\omega^{k+1}\right)$.

4. The Chan-Vese Model with Semi-Implicit Additive Operator Splitting Scheme

In this section, we present the CV model with the semi-implicit AOS scheme in biphase and multiphase cases.

4.1. Biphase Case

From equation (7), we denote:

$$f=\delta_\varepsilon(\varphi)\left\{-\left[\lambda_1\left(u_0-c_1\right)^2-\lambda_2\left(u_0-c_2\right)^2\right]-\nu\right\}, \tag{19}$$

To avoid singularities, we replace the term $|\nabla\varphi|$ with $|\nabla\varphi|_\beta=\sqrt{\varphi_x^2+\varphi_y^2+\beta}$ and denote $W=1/|\nabla\varphi|_\beta$ Discretizing (7) by employing the AOS scheme, we get the following equation:

$$\varphi^{n+1}=\frac{1}{2}\sum_{l=1}^{2}\left(I-2\Delta t A_l\left(\varphi^n\right)\right)^{-1}\hat{\varphi}^n, \tag{20}$$

The matrices A_l, for $l=1,2$, are tridiagonal matrices derived using finite differences [11] and $\hat{\varphi}^n=\varphi^n+\Delta t f$.

$$\left(A_1\left(\varphi^n\right)\varphi^{n+1}\right)_{i,j}=\mu\delta_\varepsilon\left(\varphi^n\right)\frac{E_{i+1,j}^n+E_{i,j}^n}{2h_x^2}\left(\varphi_{i+1,j}^{n+1}-\varphi_{i,j}^{n+1}\right)-\mu\delta_\varepsilon\left(\varphi^n\right)\frac{E_{i,j}^n+E_{i-1,j}^n}{2h_x^2}\left(\varphi_{i,j}^{n+1}-\varphi_{i-1,j}^{n+1}\right)$$

$$\left(A_2\left(\varphi^n\right)\varphi^{n+1}\right)_{i,j}=\mu\delta_\varepsilon\left(\varphi^n\right)\frac{E_{i,j+1}^n+E_{i,j}^n}{2h_y^2}\left(\varphi_{i,+1j}^{n+1}-\varphi_{i,j}^{n+1}\right)-\mu\delta_\varepsilon\left(\varphi^n\right)\frac{E_{i,j}^n+E_{i,j-1}^n}{2h_y^2}\left(\varphi_{i,j}^{n+1}-\varphi_{i,j-1}^{n+1}\right)$$

The algorithm of the CV model with the semi-implicit AOS in biphase case is [12]:

1) Initialize φ^0 by φ_0, $k=0$.
2) Compute f from equation (19).
3) Compute $c_1\left(\varphi^k\right)$ and $c_2\left(\varphi^k\right)$ by (5) and (6).
4) Compute φ^k using (20).
5) Check whether the solution is stationary. If not, repeat 2-5.

4.2. Multiphase Case

From equation (13), we denote:

$$f_1=\delta_\varepsilon\left(\varphi_1\right)\left\{-\left[\begin{array}{l}\left(\left(u_0-c_{11}\right)^2-\left(u_0-c_{01}\right)^2\right)\\ \left(H_\varepsilon\left(\varphi_2\right)\right)+\left(\left(u_0-c_{10}\right)^2-\left(u_0-c_{00}\right)^2\right)\\ \left(1-H_\varepsilon\left(\varphi_2\right)\right)\end{array}\right]\right\} \tag{21}$$

From equation (14), we denote:

$$f_2=\delta_\varepsilon\left(\varphi_2\right)\left\{-\left[\begin{array}{l}\left(\left(u_0-c_{11}\right)^2-\left(u_0-c_{10}\right)^2\right)\\ \left(H_\varepsilon\left(\varphi_1\right)\right)+\left(\left(u_0-c_{01}\right)^2-\left(u_0-c_{00}\right)^2\right)\\ \left(1-H_\varepsilon\left(\varphi_1\right)\right)\end{array}\right]\right\} \tag{22}$$

To avoid singularities, we replace the term $|\nabla\varphi_1|$ with $|\nabla\varphi_1|_\beta=\sqrt{\varphi_{1x}^2+\varphi_{1y}^2+\beta}$ and $|\nabla\varphi_2|$ with $|\nabla\varphi_2|_\beta=\sqrt{\varphi_{2x}^2+\varphi_{2y}^2+\beta}$

The algorithm of the CV model with the semi-implicit AOS in multiphase case is:

1) Initialize φ_1^0 and φ_2^0 by φ_{10} and φ_{20}, $k=0$
2) Compute $c_{11}\left(\varphi^k\right)$, $c_{10}\left(\varphi^k\right)$, $c_{01}\left(\varphi^k\right)$ and $c_{00}\left(\varphi^k\right)$
3) Compute f_1 and f_2 by equation (21) and (22).
4) Compute φ_1^k using (20) and φ_2^k using (20).
5) Check whether the solution is stationary. If not, repeat 2-5.

5. Experimental Results

In the biphase case, the constants are given as follow: $\nu=0$, $\Delta t=1$ and $\lambda_1=\lambda_2=1$.

In Figs. 2 - 5, we illustrate the segmentation by the CV model for boat, MR of knee, MR of brain and CT images. In Figures Figs. 6 - 9, we show the segmentation by the CV model with semi-implicite

AOS scheme for the same images. The segmentation illustrates the two phases and the results are almost similar for the two methods.

For the multiphase case, the constants are given as follow: $\nu = 0$ and $\lambda_1 = \lambda_2 = 1$. In Figs. 10 - 13, we illustrate the segmentation by the CV model for boat, MR of knee, MR of brain and CT images, but in Figs. 14 - 17, we show the segmentation by the CV model with the semi-implicite AOS scheme for the same images. The two methods give exactly the same segmentation where we can see the four phases.

The comparison study relative to time computing is summarized in Tables 1 - 4; we deduce that the CV model with semi-implicit AOS scheme reduces the time computing of the segmentation by half.

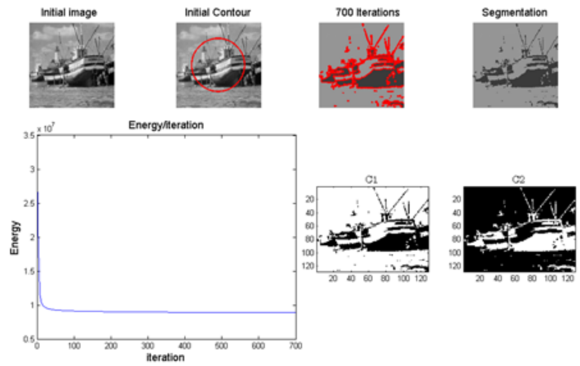

Fig. 2. Segmentation by CV model (biphase case) of boat.

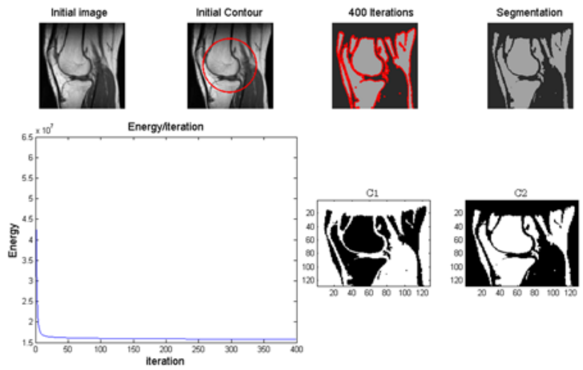

Fig. 3. Segmentation by CV model (biphase case) of MR image of knee.

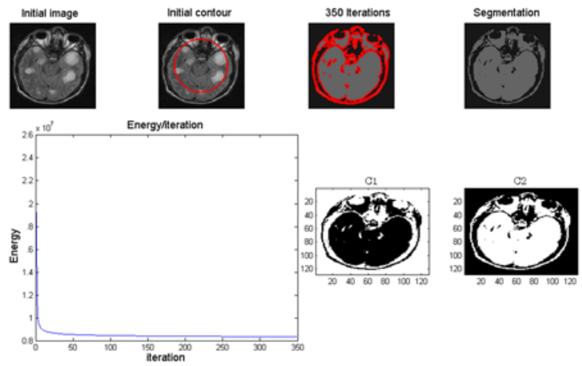

Fig. 4. Segmentation by CV model (biphase case) of MR image of brain.

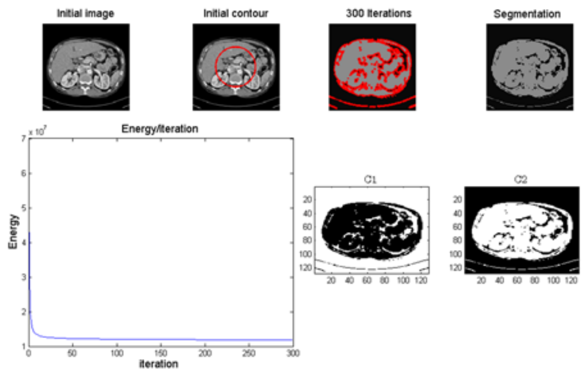

Fig. 5. Segmentation by CV model (biphase case) of CT image.

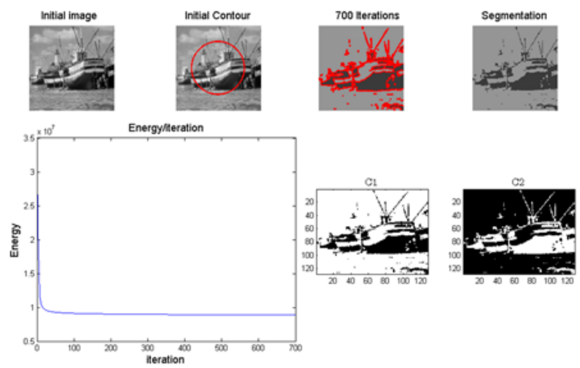

Fig. 6. Segmentation by the CV model with semi-implicit AOS scheme (biphase case) of boat.

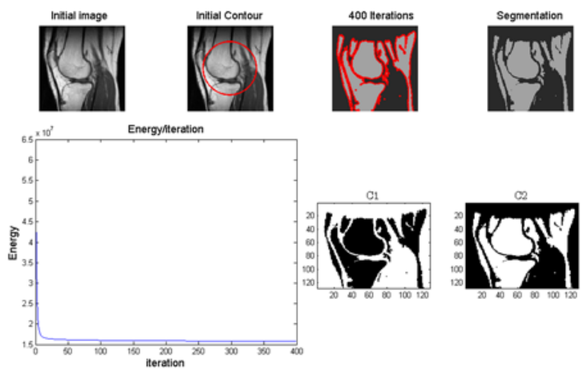

Fig. 7. Segmentation by the CV model with semi-implicit AOS scheme (biphase case) of MR image of knee.

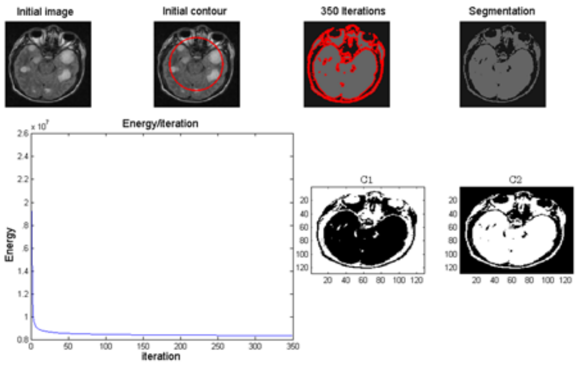

Fig. 8. Segmentation by the CV model with semi-implicit AOS scheme (biphase case) of MR image of brain.

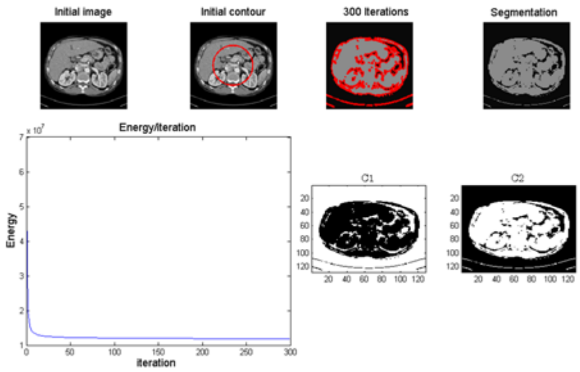

Fig. 9. Segmentation by the CV model with semi-implicit AOS scheme (biphase case) of CT image.

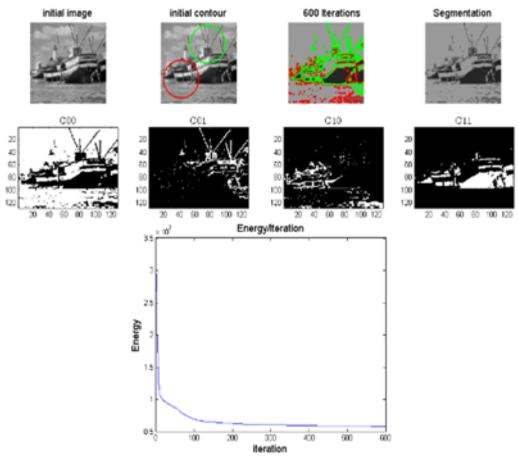

Fig. 10. Segmentation by CV model (multiphase case) of boat.

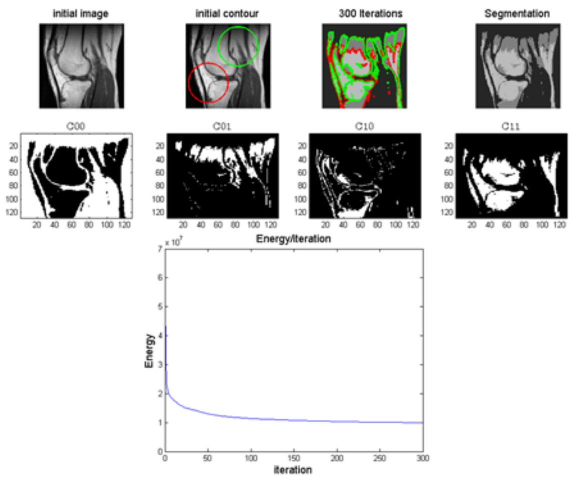

Fig. 11. Segmentation by CV model (multiphase case) of MR image of knee.

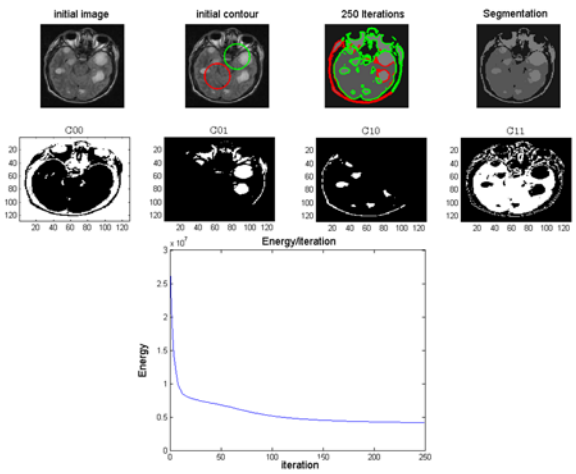

Fig. 12. Segmentation by CV model (multiphase case) of MR image of brain.

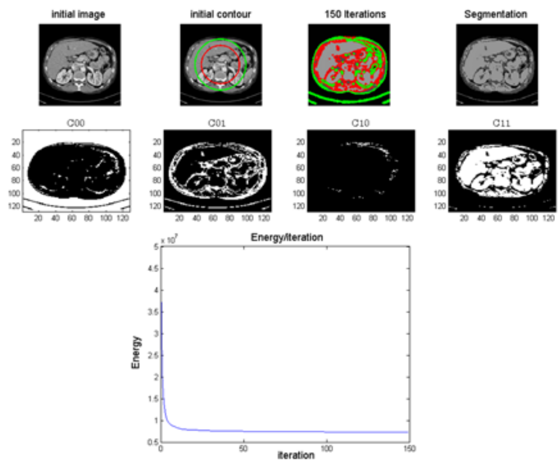

Fig. 13. Segmentation by CV model (multiphase case) of CT image.

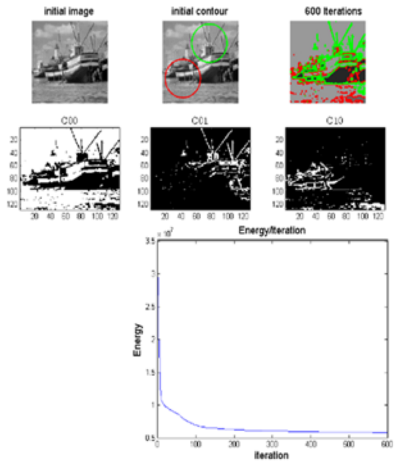

Fig. 14. Segmentation by the CV model with semi-implicit AOS scheme (multiphase case) of boat.

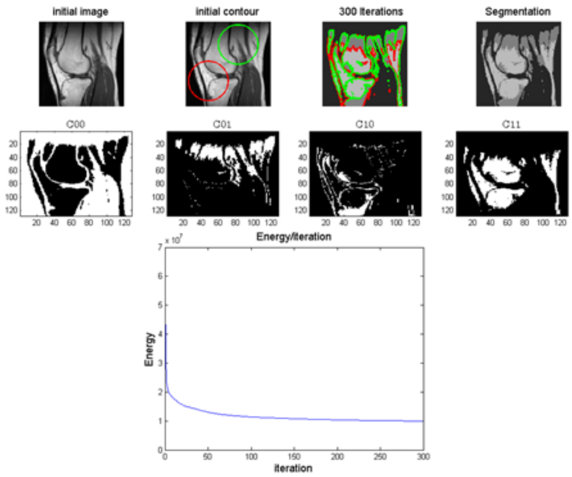

Fig. 15. Segmentation by the CV model with semi- implicit AOS scheme (multiphase case) of MR image of knee.

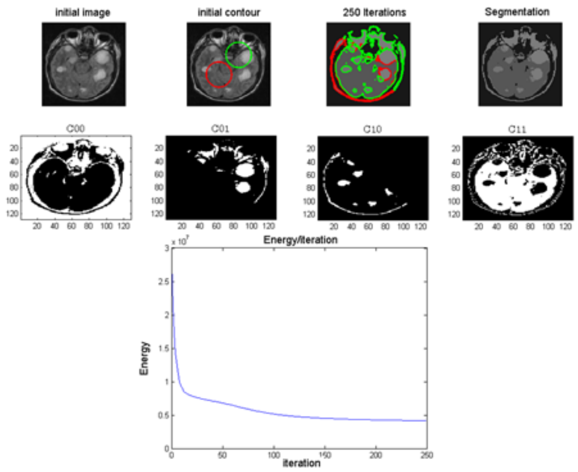

Fig. 16. Segmentation by the CV model with semi-implicit AOS scheme (multiphase case) of MR image of brain.

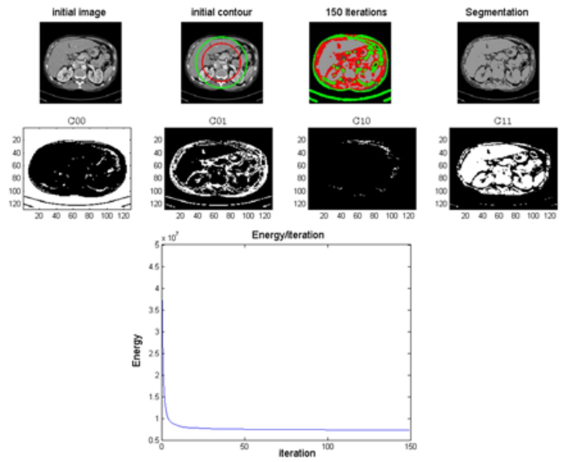

Fig. 17. Segmentation by the CV model with semi-implicit AOS scheme (multiphase case) of CT image.

Table 1. Comparison between the CV model and the CV model with semi-implicit AOS scheme of boat and MR of knee images in biphase case.

Image	boat		MR of knee	
Method	CV	CV-AOS	CV	CV-AOS
CPU time (s)	110.6671	56.7532	51.9639	22.1521

Table 2. Comparison between the CV model and the CV model with semi-implicit AOS scheme of MR of brain and CT images in biphase case.

Image	MR of brain		CT	
Method	CV	CV-AOS	CV	CV-AOS
CPU time (s)	48.8127	20.9665	48.9063	22.7761

Table 3. Comparison between the CV model and the CV model with semi-implicit AOS scheme of boat and MR of knee images in multiphase case.

Image	boat		MR of knee	
Method	CV	CV-AOS	CV	CV-AOS
CPU time (s)	158.2630	71.0429	70.3253	28.1270

Table 4. Comparison between the CV model and the CV model with semi-implicit AOS scheme of MR of brain and CT images in multiphase case.

Image	MR of brain		CT	
Method	CV	CV-AOS	CV	CV-AOS
CPU time (s)	69.9352	26.3954	39.4527	18.8449

6. Conclusions

In this paper, we have used the advantages of the semi-implicit AOS technique in order to fast the CV model for image segmentation in biphase and multiphase cases. The experimental results show that the segmentation is done in the two cases, with the superiority of the CV model with the semi-implicit scheme compared to the CV model concerning the time computing. As future work, we plan to associate the semi-implicit AOS technique with other active contour.

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